Referee's report on the habilitation thesis

Masaryk University	
Faculty	of Science
Habilitation field	Mathematics – mathematical analysis
Applicant	RNDr. Zdeněk Svoboda, CSc.
Affiliation	Masaryk University, Faculty of Science
Habilitation thesis	Asymptotic properties of functional-differential equations with delay
Referee	Dr. Sc. Astashova Irina, professor,
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The habilitation thesis presented by Zdeněk Svoboda summarizes 13 publications by the author on the asymptotic properties of functional differential equations with delays. They focus on three thematic units: Systems of linear differential equations with constant coefficients and constant delays; Application of the retract theory to the study of the asymptotic properties of functional differential equations; Study of the exponential stability of functional differential systems with delays.

The habilitation thesis divided into six Chapters.

The first Chapter serves as an introduction to the studied subject. Literature sources that the author follows are listed. In addition, some examples from different disciplines describing dynamic models with delays are mentioned to show the necessity to study differential equations with delays.

The second Chapter deals with linear systems with constant coefficients and constant delays. Auxiliary original delayed matrix functions are defined. One of them - the delayed exponential matrix is the concretization of the well-known concept of the fundamental matrix introduced for systems of first order linear equations with delay. In addition, two delayed matrix functions - the delayed matrix sine and the delayed matrix cosine are presented as analogous tools for second order linear systems with delay. The author determines the relationship between delay matrix exponential and delayed matrix sine and cosine deriving a generalization of Euler's identity for these matrices. Along with this, the possibility is discussed of using the delayed matrix exponential to solve more general linear systems of neutral type. In this chapter, the asymptotic properties of delayed matrix functions are also studied. To a given matrix, another matrix is found such that the asymptotic properties of the delayed matrix exponential of the first matrix are the same as those of the usual matrix exponential of the second one. The relationship between the above-mentioned two matrices is expressed in terms of the Lambert function.

This relationship, together with the knowledge of the eigenvalues of the given matrix, is used to determine the asymptotic properties of the delayed matrix exponential. A theorem is proved describing this relationship illustrated by examples. Using the generalization of Euler's identity, the boundlessness of the spectral norm of the delayed matrix sine and the delayed matrix cosine is demonstrated.
The third Chapter discusses the so-called retract method having its origin in the papers by Wazewski where this method was built for ordinary differential equations. Later, Rybakowski successfully enlarged the Wazewski topological method to functional differential equations with bounded delays. This generalization uses the concept of a system of curves with several prescribed properties. The contribution of the candidate in this chapter is a non-trivial modification of the retract method for neutral-type delayed equations. The progress was achieved by redefining the polyfacial sets with respect to delayed differential equations of neutral type, supplying a new definition of a polyfacial set regular with respect to the neutral equations and systems of subsidiary inequalities and by formulating the related topological principle.

The following fourth and fifth Chapters discuss the use of the topological method in some particular cases of the general non-linear case discussed in the previous chapter. Here, the method is also described of asymptotic integration of some classes of equations and asymptotic decompositions of their solutions are found. An interesting example of a substantial dependence of the asymptotic decomposition of the solution on the character of delay is presented.

The fourth Chapter ends with criteria guaranteeing the existence of positive solutions of non-linear equations for $t \to \infty$.

The fifth Chapter summarizes the new criteria for the existence of positive solutions of linear equations with delay. First, general sufficient and necessary conditions of existence of positive solutions of linear systems of delayed differential equations are given. Moreover, sufficient conditions for the existence of positive solutions are also derived. The chapter is structured so that it deals separately with equations with discrete delays and with equations with distributed delays. Particular attention is paid to the existence of a positive solution of the scalar equation

$$\dot{y}(t) = -a(t)y(t - \tau(t))$$

in the critical case. Two ways of generalizing the well-known results in the case of constant delay are suggested and new results are derived. Further, the scalar equation with distributed delay is studied and, in this case, new criteria for the existence of positive solutions are derived as well.

The last part of this chapter deals with an application of the retract principle to neutral differential equations. Here, a criterion is given for the existence of positive decreasing solutions of the scalar neutral equations

$$\dot{y}(t) = -c(t)(y(t - \tau(t))) + d(t)\dot{y}(t - \delta(t)).$$

The conditions for the existence of positive decreasing solutions are then applied to the case of the equation with constant coefficients. It is shown that the derived criteria generalize the well-known previous criteria. The originality and significance of the results in this chapter is substantiated by numerous citations of results by authors working in the field.
The last chapter is devoted to the study of uniform exponential stability of the linear system with variable delays

$$\dot{x}_i(t) = -\sum_{j=1}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k(t) x_j(h_{ij}^k(t)), \quad i = 1, \dots, m.$$

The original results, generalizing the previously published ones, are obtained. The topic of thesis is important and interesting with many important results derived by the candidate. The main results are new, original and non-improvable, and have many applications.

To summarize my review, I assess the scientific results achieved by RNDr. Zdeněk Svoboda, CSc. as being of an exceptional importance in qualitative theory of differential equations and has discovered new approaches in theory of delayed differential equations.

Referee's questions:

1. Are there ways for further progress in the field?
2. Is it possible to prove similar results for delayed differential equations if delays are unbounded and with infinite memory?

Conclusion

Habilitation thesis by Zdeněk Svoboda entitled "Asymptotic properties of functional-differential equations with delay" **meets** the standard requirements for habilitation theses in mathematical analysis.

In September, 2018, date 12

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