Referee's report on the habilitation thesis

Masaryk University	
Faculty	of Science
Habilitation field	Mathematics – mathematical analysis
Applicant	RNDr. Zdeněk Svoboda, CSc.
Affiliation	Masaryk University, Faculty of Science
Habilitation thesis	Asymptotic properties of functional-differential equations with delay
Referee	Oleksandr Boichuk
Affiliation	doctor of physical and mathematical sciences, professor, corresponding member of the National Academy of Sciences of Ukraine, head of the laboratory boundary-value problems of differential equations of the Institute of Mathematics of the National Academy of Sciences of Ukraine

Text of the report (the extent according to the referee's opinion) Habilitation Tesis "Asymptotic Properties Functional-Differential Equations with Delay" of Zdenek Svoboda

The paper is devoted to actual problems of investigating the qualitative behavior of solutions of differential-functional equations, in particular, solutions of equations with delay. The work consists of 50 pages, and is based on thirteen works, listed in the "List of commented research papers".

It consists of six sections.
The first section (pages 2-4) is an introductory, it briefly contains examples of the use of the apparatus of differential equations with time-delay, when modeling the dynamics of processes.
The second section (pages 5-15) is devoted to systems of linear differential equations with constant coefficients and with time-delay argument. At the beginning, systems of the first order are considered. It is shown, that if for the system without delay the fundamental matrix of solutions is a matrix exponential, which is a matrix series, then for systems with "pure delay" the analogue will be a retarded matrix exponential representing as a finite matrix sum of a special kind "similar" to the matrix exponential. Next, second order systems with "pure delay" are considered, the general solution is written using special matrix functions called "retarded matrix cosines and retarded matrix sinus". The connection between matrix exponentials and retarded matrix sinus and cosines is shown.
Of interest is the development of the integral representation of the solution of the Cauchy problem for second-order systems with two delays of neutral type and for second-order systems of a special form containing the first derivative with commuting matrices. For systems of this type, a solution of the Cauchy problem in the analytic form is written (Theorems 1, 2, 3).
The author considers one of the weakly investigated problems - the connection between the Lambert function and the equations with delay. As is known, the left-hand side of the characteristic equation of a linear stationary system without delay is a polynomial and has a

number of roots equal to the order of the polynomial. The characteristic equation of a differential equation with a pure delay, of the first order

$$\dot{x}(t) = ax(t-\tau)$$

has the form of a quasipolynomial

$$\lambda = ae^{-\lambda\tau} \text{ or } a = \lambda e^{\lambda\tau}$$

and has a countable number of roots, which it is not possible to compute. Therefore, it is of interest to obtain asymptotic estimates of their behavior. This problem is analogous to the investigation of the behavior of the Lambert function, which is the inverse of the function

$$f(w) = ze^{w}.$$

The author obtained a number of statements concerning the distribution of the roots of the characteristic equation. The exponential estimates of the matrix $e_{\tau}^{Bk\tau}$, $k \to \infty$ (2.35) is obtained. In Theorem 6, when the condition is satisfied, a series of estimates of the norms of the matrix delayed exponential is obtained. The results are shown in Fig. 2.1, 2.2. “Model systems” (2.46), (2.47) are considered. In conclusion of this section, estimates are obtained for the growth of the norms of the delayed matrix sine and matrix cosine

In the third section (pp. 16-23) topological methods for investigating functional-differential equations are considered. Using the retractor method, existence theorems are proved. The use of the retractor principle for neutral functional differential equations is considered (Theorems 8, 9, p. 20).

The fourth section (pages 24-31) is devoted to the application of the results presented above to certain particular cases of nonlinear equations. It should be noted that effective results for nonlinear systems can be obtained only for systems of a special kind. We consider nonlinear systems with a distinguished linear part of the diagonal form and a nonlinearity of the form "product of phase coordinates with aftereffect". Estimates of solutions (Theorems 12, 13) for systems with nonlinearities of a "special type" are obtained. They have the form of theorems of the type of "comparison theorems"

An interesting example (Section 4.2, pp. 28-29)

$$\dot{y}(t) = -\cos(tu(\xi(t)))$$

with an emphasis on the linear part and the expansion of the right-hand side in a series is considered as an application. An approximate solution is obtained, represented in the form of an expansion in a series (p. 29).

In the last subsection 4.3, conditions for the existence of positive solutions of nonlinear systems are considered

In the fifth section (pp. 32-40), the problem of the existence of positive solutions in linear systems is considered. General results for nonlinear systems are preliminarily considered (Theorems 17, 18). Next, "standard linear homogeneous systems" (5.5) is considered. Conditions are obtained under certain conditions imposed on the coefficients of the linear part of the matrices of the current coordinates with delay and without delay, conditions for the existence of positive solutions are obtained (Theorems 19, 20). Linear scalar equations with aftereffect are studied in detail. The equation with one delay (5.14) is preliminarily considered. An example is given (5.16), in which conditions for the existence of a positive solution are demonstrated.

The questions of the existence of positive solutions of a scalar equation with "special non-autonomy" are considered (Theorem 26)

$$\dot{y}(t) = -a(t)y(t-\tau(t)),\ a(t) \le \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^3} + \dots$$

The conditions for the positivity of the solution are obtained under constant time-delay. They have the form formulated in Theorems 26, 27.

Similar problems are considered for linear equations of neutral type

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)\dot{y}(t-\delta(t)),$$

where $c(t)$, $d(t)$ - non-negative functions $\tau(t)$, $\delta(t)$ - positive, bounded functions (Theorem 33). A special case of constant functions and, correspondingly, equations of the form

$$\dot{y}(t) = -cy(t-\tau(t)) + d\dot{y}(t-\delta(t)),$$

is considered.

In this case, the condition for the positivity of the solutions is the existence of a positive constant λ for which inequality

$$\lambda \geq ce^{\lambda\tau} + \lambda de\lambda\delta.$$

is satisfied.

The sixth section (pages 41-44) deals with exponential stability problems for linear systems with delay

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}x(t-\tau_{ij})$$

The conditions are expressed as constraints on the matrix coefficients and the delay values and are formulated in Theorems 1-3. The obtained conditions are a generalization of known stability conditions for one equation with one delay and are formulated in Theorems 1-3. The general result is given in Theorem 35.

Referee's questions (the number of queries according to the referee's opinion)

1. Theorem 4 mentions a matrix A. At the same time, in expression (2.29), the matrix B is used? Expressions $W_0(\lambda_i, \tau)$, $i = \overline{1, n}$ are not defined in (2.30), (2.31)

2. The dependence of the system (5.10) is incomprehensible. In particular, the entry of an argument with time-delay.

3. In the sixth section, the numbering of theorems is violated.

Conclusion

Habilitation thesis by Zdenek Svoboda entitled "Asymptotic properties of functional-differential equations with delay" **meets** - the standard requirements for habilitation theses in mathematical analysis.

In Kyiv, Ukraine, date 10.05.20: