Referee's report on the habilitation thesis

Masaryk University	
Faculty	of Science
Habilitation field	Mathematics – mathematical analysis
Applicant	RNDr. Zdeněk Svoboda, CSc.
Affiliation	Masaryk University, Faculty of Science
Habilitation thesis	Asymptotic properties of functional-differential equations with delay

Referee:
Prof Alexander Domoshnitsky
Affiliation:
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Text of the report (the extent according to the referee's opinion)

The thesis is concerned with the asymptotic properties of functional differential equations with delay and is divided into six Chapters. In the first Chapter (Introduction), the author illustrates the importance of delayed functional-differential equations using some examples modelling real-world phenomena of different natures.

In the second Chapter, linear systems with constant coefficients and constant delays are studied. To formally describe their solutions, some special matrix functions called delayed matrix exponential, delayed matrix sine, and delayed matrix cosine are used as a specification of the previously defined fundamental matrix. The author shows the relation between the delayed matrix exponential and the delayed matrix sine and cosine. The resulting formula is a generalization of the well-known Euler's identity. Moreover, with the use of these special matrix functions, representations are derived of the solutions of second-order linear delayed systems with constant coefficients. In this connection, an important problem is to clarify the asymptotic properties of delayed matrix functions. In this part, the following result is derived – a matrix which is the limit of the matrix "ratio" from the right and from the left of delayed matrix exponential functions, computed at nodes tending to infinity, equals to a constant matrix. To get this result, non-trivial properties of the Lambert function must be used. Interesting consequences are shown at the end of this chapter. Among others, the generalized Euler identity is used to demonstrate the unboundedness of the norms of the delayed matrix sine and cosine.

In the second Chapter, a topological method introduced by T. Wazewski for ordinary differential equations and later successfully modified by K. Rybakowski for delayed differential equations, is re-formulated for the case of delayed functional differential equations with un-bounded delays and with finite memory. This progress has been achieved by utilizing the concept of p-type retarded functional differential equations in the setting of Lakshmikantham, L.Wen and B. Zhang. Moreover, using a technique of polyfacial sets, regular polyfacial sets, and some auxiliary inequalities, the retract method has been extended to neutral p-type retarded functional differential equations.

In Chapter 4, the retract method is applied to the asymptotic integration of perturbed linear systems where perturbations have the form of an infinite series with terms depending on various delayed arguments. The form of such perturbations has been inspired by the first Lyapunov method where solutions are constructed as convergent power series. In the case

considered, solutions are constructed in the form of formal power series (where the general solution of the related homogeneous linear system without perturbations is powered). The topological method is then used to prove the existence of solutions of the original equation in the domain determined by partial sums of the formal power series. By this method, the substantial differences of the asymptotic expansions of solutions for two cases of different delays (bounded and un-bounded) are demonstrated using the equation with one delay

$\dot{y}(t) = -\cos(ty(t-\tau(t)))$

Among others, it is shown that, for a bounded delay, asymptotic decompositions of solutions are asymptotically the same as the solutions of the equation without delay and, for an un-bounded delay, the asymptotic decomposition of the solutions is similar to the asymptotic decomposition of the equation with the perturbation term being equal to the first term of the perturbed series. The second part of this chapter brings criteria guaranteeing the existence of a positive solution of p-type retarded functional differential non-linear systems.

Chapter 5 develops new criteria for the existence of positive solutions to general functional delayed linear systems of equations. Both the criteria (of the "if and only if" type) and the sufficient conditions are derived. Special attention is paid to the investigation of the critical scalar case. Comparisons with the results of other authors previously studying this problem are given. The results derived are original and generalize some of the previous ones. This chapter is finished with a criterion for the existence of positive decreasing solutions to the scalar delayed neutral equation

$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)\dot{y}(t-\delta(t))$.

The novelty of the results is obvious since a positive solution is defined as a continuous and continuously differentiable function and the results generalize the previously known criteria.

The last Chapter 6 studies the uniform exponential stability of the solutions to the system

$$\dot{x}(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^{k}(t)x_j(h_{ij}^{k}(t)), \qquad i = 1,\ldots,m.$$

The results obtained are new and independent of the results of other authors.

The results achieved by Zdenek Svoboda have been published in quality mathematical journals and have had good response judged by the numerous citations in papers by other authors, the invitations to serve as a referee of submitted papers to editorial boards of various mathematical journals, or to deliver talk at conferences. The habilitation thesis meets all the criteria relevant for this thesis type. The applicant deserves the scientific-pedagogical degree of "docent". I assess the habilitation thesis very positively and recommend it for defence.

Several minor remarks:

Referee's questions (the number of queries according to the referee's opinion)

1. p.39 I think English in the Corollary 3 on p. 39 should be fixed:
Corollary 3. For the existence of a positive decreasing solution of (5.37) on [to-r,infinity),
the existence is sufficient of a positive constant l such that inequality

2. In the parts of the thesis, where the Lambert function is used, the author proves several interesting results, and the text is written very clearly.
Only in the relevant introduction page (page 5) there are a few small inaccuracies in the text:

- The sentence

"Our results are stated in theorems of this chapter are proved .."
Should be
"Our results stated in theorems of this chapter are proved .."
Or
"Our results that are stated in theorems of this chapter are proved .."

- The sentence

"The application … has recently been in the case of …"
Should be
"The application … has recently been done in the case of …"

- The sentence

"The papers [59], [a6] studies the asymptotic properties … "
Should be
"The papers [59], [a6] study the asymptotic properties … "

3. In the chapters about representation of solutions and about stability of delay systems, the book
N. V. Azbelev, P. M. Simonov, Stability of differential equations with aftereffect. Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003,
has to be noted.

Conclusion

Habilitation thesis by Zdenek Svoboda entitled " **Asymptotic properties of functional-differential equations with delay**"
meets the standard requirements for habilitation theses in mathematical analysis.

Ariel University, 1.6.2018