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In this report, I give an evaluation of the habilitation thesis by Zuzana Pátíková entitled "Riccati methods for half-linear differential equations".

This thesis is mainly concerned with the qualitative theory of half-linear differential equations. In particular, the author focuses on the oscillation problem for Sturm-Liouville type half-linear equations and discusses the conditional oscillation of such equations. As for the main analysis method, the author often transforms Sturm-Liouville type differential equations into Riccati type differential equations. This method is the so-called Riccati technique, which is a method of reducing second-order equations to first-order equations, and it is very useful for studying Sturm-Liouville type equations. The author has given several results by applying the Riccati technique to various types of equations.

In the following, I comment on each chapter of this thesis.

Chapter 1 covers the scientific background for the Sturm-Liouville type half-linear equations and some preparations. In particular, the author discusses the importance of half-linear differential equations, several applications, domestic and overseas trends related to her research, and the positioning of her research in the relevant field. I think it is very interesting. When I read this chapter, I remembered staying in Brno about 10 years ago and having a mathematical discussion with professor Došlý and with his collaborators.

In Chapter 2, the author considers half-linear differential equations including the Riemann-Weber type equation. By using regularly varying functions, the author characterizes solutions of the equations. Moreover, the author extends the asymptotic formula given by Elbert and Schneider [58]. In Chapter 3, the author gives Hille-Nehari type (non)oscillation criteria and Hille-Wintner type comparison theorems. In these chapters, the Riccati technique is used and its usefulness can be confirmed.

In Chapter 4, the author considers the oscillation problem for neutral half-linear differential equations and has shown that the Riccati technique is also useful for this problem. Hence, it turns out that the Riccati technique can be used for various types of equations. Note that, in this chapter, the author does not discuss the existence and uniqueness of solutions to the initial value problem.

In order to discuss the oscillation problem, I think that it would be better to assume that the existence and uniqueness of solutions are guaranteed for the initial value problem.

In Chapter 5, Sturm-Liouville type half-linear equations are studied by numerical approach. Although the finite difference method is often used in the numerical approach of differential equations, as another approach, the author uses the differential transformation method, which focuses on coefficients of the Taylor series. By using this method, half-linear differential equations are transformed into difference equations. Since I have seen such a transformation for the first time, I feel that it is a new direction of research for half-linear equations.

The author also suggests some open problems and research directions. I hope she will tackle these problems and solve them in the future. In particular, I would like her to challenge whether the Riccati technique is useful for differential equations with $p(t)$-Laplacian.

This thesis has been reconstructed based on eight original papers submitted as related research. Most of the original papers have been published in the last few years, and therefore her recent research activities are extremely active. In these papers, by using the Riccati technique and the differential transformation method, Sturm-Liouville type half-linear equations are considered, so I think that she has good talents for not only mathematical analysis but also numerical analysis. In addition, this thesis is easy to read and there are almost no typos. Moreover, as far as I know, the results presented in this thesis are new and original.

In conclusion, I can judge that this thesis by Zuzana Pátíková entitled "Riccati methods for half-linear differential equations" is of high quality and meets the standard requirements for a habilitation thesis in the research field of mathematical analysis.

Finally, I suggest some minor remarks:

- Delete "(t)" on page 8, line 5.
- Change "$C(t) - c(t)$" to "$c(t) - \tilde{c}(t)$" on page 12, line 2.
- Insert "h" between "function" and "such" on page 24, line 20.
- Change "where" to "were" on page 41, line 9.

Yours sincerely,

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