MASARYK UNIVERSITY
FACULTY OF SCIENCE

QUALITATIVE THEORY OF FRACTIONAL DIFFERENTIAL SYSTEMS WITH TIME DELAY

HABILITATION THESIS

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Preface

The idea behind derivatives of non-integer orders traces back nearly as far as the classical derivatives themselves. What began in the 17th century as a puzzling thought exercise has evolved into one of the most influential mathematical fields of recent decades, known as fractional calculus. My background in both mathematics and physics continues to feed my interest in this elegant discipline and its impact on both theory and applications.

In the realm of applied sciences, fractional calculus has gained recognition for its linear descriptions of complex systems characterized by nonlocal or memory-based behaviour, traditionally treated within the nonlinear domain. Theoretically, the ability to continuously transition between derivative orders reveals previously unseen connections and enables the study of various phenomena. However, the generalizing nature of fractional calculus also poses a risk: researchers are tempted to deal with artificial, easily solvable problems that neither enrich applications nor contribute much to theory, resulting in mere formalism. With this in mind, I have made it my goal to focus on key problems that help to shed light on deeper mathematical principles and behaviour of complex systems.

This habilitation thesis combines the key results of my seven selected papers [6, 10–13, 15, 37] from 2016-2023. Their unifying theme is the qualitative analysis of fractional delay differential equations, a class of mathematical models involving fractional derivatives and time delays representing inherent lags in the system. The key aim is two-fold: first, to better understand, predict and control the behaviour of such systems, which is essential for numerous applications ranging from physics and biology to engineering and finance. Second, to integrate fractional calculus into the broader landscape of dynamic systems theory, where it can illuminate the intricate interplay of delayed responses and memory effects in shaping system behaviour.

The thesis comprises four chapters that present the main results and provide commentary on key proofs referring the respective papers, and seven appendices containing the full text of the selected published papers. Chapter 1 provides a context of classical and fractional qualitative analysis and introduces known limit cases of our study. Chapter 2 summarizes the original results on one-term fractional delay differential systems. It sets the theoretical foundation for subsequent work and presents key stability and oscillatory conditions, often in non-improvable form. Chapter 3 focuses on two-term fractional delay differential equations. It explores the relationships between derivative order and the stability regions for the system's coefficients. Finally, Chapter 4 offers concluding remarks and reflections on this

research.

I express deep gratitude to Jan Čermák, my former advisor, leader of the scientific group I am proud to be part of and my most frequent co-author. His insights, leadership, continuous support and willingness to share his broad mathematical knowledge were essential for the completion of this work. I also thank my other co-authors and colleagues for their collaboration, namely Zuzana Došlá, Jan Horníček and Luděk Nechvátal who participated in some of the seven papers forming the base of this thesis. I extend my thanks to Alberto Cabada and Matej Dolník who influenced my thinking during joint work on other topics. In addition to the professional support I have received, I am especially grateful to Lída Kiselová, my dear wife, and to my children Bára, Magda and Teodor for their endless patience, encouragement and understanding without which I could not have accomplished this work. I thank my parents, grandparents and entire family as well as my friends for an occasional gentle push that kept me moving forward.

Brno, September 2024 Tomáš Kisela

Contents

Chapter 1

Wider context: classical and fractional qualitative analysis

The study of dynamic systems involving time delays is a classical area of mathematical analysis with significant real-world applications. These systems can accurately model processes that do not respond instantaneously to changes in their state or environment, such as biological systems, economic models, and engineering processes. Despite extensive research efforts, many questions related to the stability and control of these classical systems remain unanswered, primarily due to the inherent challenges of incorporating delayed responses (see, e.g. [21, 22, 38]).

Fractional calculus (extending the concepts of differentiation and integration to non-integer orders) is known for its ability to address the complexities of dynamic systems with nonlocal and memory effects, such as systems where the future state depends on a continuum of past states. The corresponding, so-called fractional, dynamic systems attract a significant attention of scientific community for several decades (see, e.g. [25, 26, 32, 48, 50, 53]). Many qualitative results of classical calculus already found their fractional counterparts (in particular in linear case), many wait for further progress and many may be impossible to generalize.

The developing interest in fractional calculus among scientists and engineers is largely due to its applications and further potential in control theory. Thus, the need to model the fractional systems with delayed feedback with sufficient precision is growing (see [20, 27, 49, 52]). In particular, since there appears to be a clash of two forces: while fractional systems of lower orders typically show larger stability regions, growing delay tends to destabilize the system. That is why this thesis focuses on the domain connecting fractional derivatives and time delays, on the qualitative theory of linear fractional delay differential systems (FDDS). Study of FDDS serves, besides its theoretical value, as a mathematical basis for effective feedback control of complex systems with memory.

This chapter sets the stage by providing the necessary context. We recall basic notions of fractional calculus and qualitative theory, present some classical results serving as comparisons later in the text and outline the basic ideas behind qualitative analysis.

1.1 Basic notions

As mentioned above, the subject of this thesis is a study of systems involving fractional derivative. Throughout the text we utilize the following definitions: Let a be a real number and let f be a real scalar function defined on (a,∞). Its fractional integral of positive real order γ is given by

$$\mathrm{D}_a^{-\gamma} f(t) = \int_a^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} f(\xi)\mathrm{d}\xi, \qquad t \in (a,\infty)\,.$$

The (Caputo) fractional derivative of positive real order α is given by

$$\mathrm{D}_a^{\alpha} f(t) = \mathrm{D}_a^{-(\lceil\alpha\rceil-\alpha)} \left(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} f(t)\right), \qquad t \in (a,\infty) \tag{1.1}$$

where $\lceil\cdot\rceil$ denotes an upper integer part (so-called ceiling function). As it is customary, we put $\mathrm{D}_a^0 f(t) = f(t)$. Besides the Caputo definition (1.1) employed throughout this thesis, some authors use the Riemann-Liouville derivative applying the fractional integral before the integer-order derivative. We mention this approach only occasionally for comparison. For more basics of fractional calculus we refer to [32,53].

If f is a vector function, the corresponding fractional operators are considered component-wise, if f is a complex-valued function, the corresponding fractional operators are introduced for its real and imaginary part separately.

The terminology that we employ is based on the classical qualitative theory:

- A linear differential system is said to be *stable* if all its solutions are bounded as $t \to \infty$.
- A linear differential system is said to be *asymptotically stable* if all its solutions tend to zero as $t \to \infty$.
- The set of all parameters' values for which the differential system is asymptotically stable is called the *stability region*.
- The solution to a differential system is called *oscillatory* if its set of zeros is unbounded.

For more precise large-time solution descriptions we use the following asymptotic notations (K being a suitable positive real):

$$f \sim g \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{f(t)}{Kg(t)} = 1\,,$$

$$f \sim_{sup} g \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{f(t)}{Kg(t)} = 1\,,$$

$$f = \mathcal{O}(g) \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{|f(t)|}{g(t)} < \infty\,,$$

$$f = o(g) \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{f(t)}{g(t)} = 0\,.$$

In addition to the *asymptotic equivalence* $\sim$, they enable us to describe asymptotics of a wider class of functions (in particular, unbounded oscillatory functions).

1.2 Classical results of qualitative analysis

In this section, we summarize the most important results regarding qualitative properties of fractional differential systems without delay, ordinary delay differential systems and ordinary differential systems with both delayed and undelayed terms. In particular, we recall the inequalities defining stability regions for the corresponding problems. We also provide commentary to other relevant sources expanding these results.

Fractional differential systems without delay

Let us consider the system

$$\mathrm{D}_0^\alpha y(t) = Ay(t)\,, \quad t \in [0, \infty) \tag{1.2}$$

where A is a constant real $d \times d$ matrix and $\alpha > 0$. The classic stability result comes from [45] and corresponds to the following assertion.

Theorem 1.1. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and let λ_j $(j = 1, 2, \ldots, d)$ be all eigenvalues of A. Then (1.2) is asymptotically stable if and only if*

$$0 < \alpha < 2 \qquad \text{and} \qquad |\mathrm{Arg}\,(\lambda_j)| > \alpha\pi/2 \qquad \text{for all } j\,.$$

Moreover, any solution y tends to zero algebraically as $y \sim t^{-\alpha}$ as $t \to \infty$.

Corollary 1.2. *The stability region of (1.2) is given by*

$$\mathcal{S}_\alpha = \{\lambda \in \mathbb{C} : |\mathrm{Arg}\,(\lambda)| > \alpha\pi/2\}\,, \quad 0 < \alpha < 2\,.$$

Remark 1.3. The proof was given in [45] and the used proving technique enables to show several other assertions:
a) Originally, only the order less than one was considered, however, the technique works analogously for higher orders.
b) It was also proven that (1.2) is stable if and only if $0 < \alpha < 2$ and $|\mathrm{Arg}\,(\lambda_j)| \geq \alpha\pi/2$ for all j and those eigenvalues with the principal argument equaling to $\pm\alpha\pi/2$ have geometric multiplicity one.
c) Unbounded solutions of (1.2) follow the asymptotic relation $y \sim_{sup} t^k \exp(\lambda^{1/\alpha} t)$ where k is a suitable nonnegative integer.
d) All solutions of (1.2) tending to zero as $t \to \infty$ are non-oscillatory. For other solutions, oscillatory property might occur.

Figures 1.1 and 1.2 illustrate the evolution of the stability region $\mathcal{S}_\alpha$ for increasing α. We note that for $\alpha \to 1$ the stability boundary coincides with imaginary axis and for $\alpha \to 2^-$ the stability region degenerates into an empty set. That shows the agreement with the classical theory for both first and second order differential systems.

In case of the scalar equation, i.e. $\mathrm{D}_0^\alpha x(t) = \lambda x(t)$ with λ being real, the asymptotic stability and stability occur for $\lambda < 0$ and $\lambda \leq 0$, respectively (provided $\alpha \in (0, 2)$).

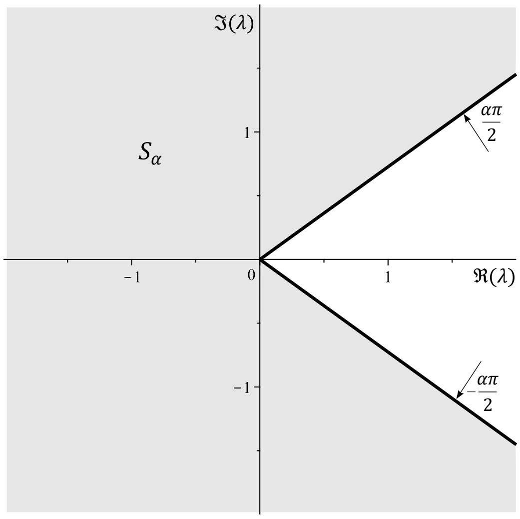

Figure 1.1: Stability region $\mathcal{S}_\alpha$ for (1.2) with $\alpha = 0.4 < 1$.

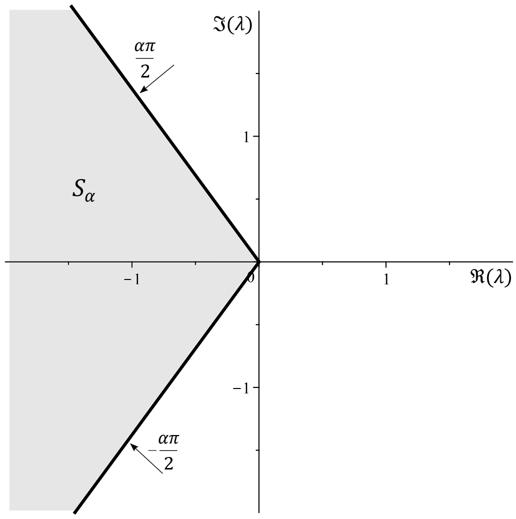

Figure 1.2: Stability region $\mathcal{S}_\alpha$ for (1.2) with $\alpha = 1.4 > 1$.

First-order delay differential system

Let us consider the system with time delay

$$y'(t) = Ay(t-\tau), \qquad t \in [0,\infty), \tag{1.3}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag. Its main stability and asymptotic properties were derived in, e.g. [23] and can be formulated as

Theorem 1.4. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and let λ_j $(j = 1, 2, \dots, d)$ be all eigenvalues of A. Then (1.3) is asymptotically stable if and only if*

$$\tau|\lambda_j| < |\mathrm{Arg}\,(\lambda_j)| - \pi/2 \qquad \textit{for all } j\,.$$

Moreover, any solution y tends to zero exponentially as $t \to \infty$.

Corollary 1.5. *The stability region of (1.3) is given by*

$$\mathcal{S}_1^\tau = \left\{\lambda \in \mathbb{C} : |\lambda| < \frac{|\mathrm{Arg}\,(\lambda)| - \pi/2}{\tau}\,,\ |\mathrm{Arg}\,(\lambda)| > \frac{\pi}{2}\right\}, \quad \tau > 0\,.$$

Oscillation properties of (1.3) can be written as follows (see, e.g. [21]).

Theorem 1.6. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and let λ_j $(j = 1, 2, \dots, d)$ be all eigenvalues of A. Then all solutions of* (1.3) *oscillate if and only if*

$$\lambda_j \in \mathbb{C} \setminus [-1/(\tau e), \infty) \qquad \textit{for all } j\,,$$

i.e. A has no real eigenvalues in $[-1/(\tau e), \infty)$.

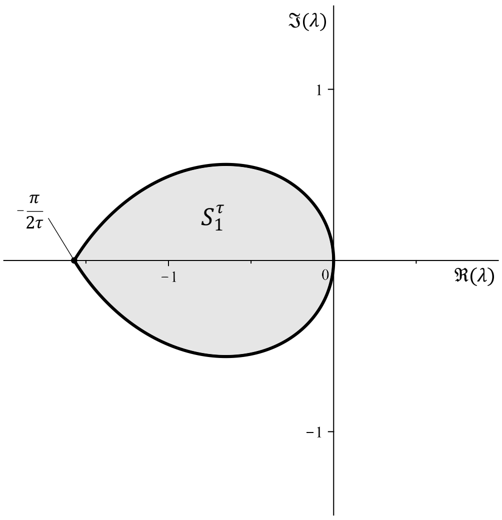

Figure 1.3: Stability region $\mathcal{S}_1^\tau$ for (1.3) depicted for the value $\tau = 1$.

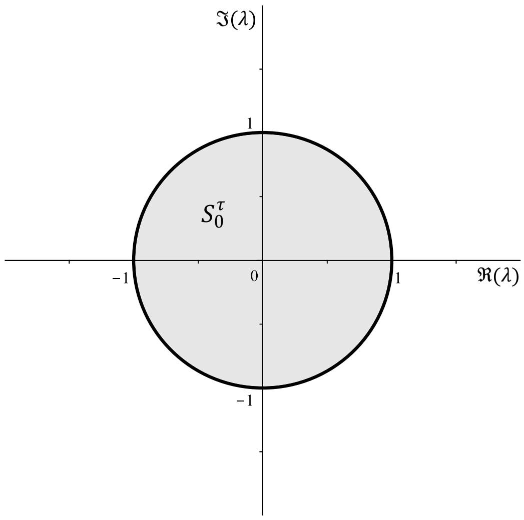

Figure 1.4: Stability region $\mathcal{S}_0^\tau$ for (1.4), independent of τ.

Figure 1.3 shows the stability region for (1.3), highlighting that in the case of scalar equation $x'(t) = \lambda x(t-\tau)$ with λ being real, the asymptotic stability condition reduces to $-\pi/(2\tau) < \lambda < 0$. Also, in scalar case all solutions oscillate if and only if $\lambda < -1/(\tau \mathrm{e})$.

Discrete system

Let us consider the discrete system

$$y(n) = Ay(n-\tau), \qquad t \in \{\tau, 2\tau \dots\}, \tag{1.4}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag. This system can be viewed as a modification of (1.3) where the derivative is removed (the derivative order is changed to zero) and the time is discretized into multiples of τ. It is well-known that the corresponding stability region (see Figure 1.4) is given by a unit circle with no dependence on the value of τ, i.e.

$$\mathcal{S}_0^\tau = \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \quad \tau > 0.$$

First-order differential equation with both delayed and undelayed terms

If an undelayed term is added to the right-hand side of (1.3), we obtain a system for which the stability analysis is quite difficult even in the planar case. Corresponding necessary and sufficient stability conditions given in terms of system parameters are known only in very special cases, e.g. if A, B are simultaneously triangularizable. For more details see [4, 31, 46]. Hence, regarding right-hand side composed of both delayed and undelayed terms, we will focus on scalar equations.

Let us consider the equation

$$y'(t) = ay(t) + by(t-\tau), \qquad t \in [0, \infty), \tag{1.5}$$

where a, b are real and $\tau > 0$ is a constant real lag. Its stability properties can be written as (see [24])

Theorem 1.7. *Let $a, b \in \mathbb{R}$, $\tau \in \mathbb{R}^+$. Then (1.5) is asymptotically stable if and only if either*

$$a \leq b < -a \qquad \textit{and} \qquad \tau \textit{ is arbitrary},$$

or

$$|a| + b < 0 \qquad \textit{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}.$$

Corollary 1.8. *The stability region of (1.5) is given by*

$$\mathcal{S}_1^\tau = \big\{(a, b) \in \mathbb{R}^2 : a - b \leq 0 \quad \textit{and} \quad a + b < 0\big\} \\ \cup \left\{(a, b) \in \mathbb{R}^2 : |a| + b < 0 \quad \textit{and} \quad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}\right\}.$$

Figure 1.5 displays the stability region for (1.5) in the (a, b)–plane. The top part of the stability boundary is formed by the axis of the second and fourth quadrants corresponding to the first condition in Theorem 1.7. The bottom part of the stability boundary representing the second condition in Theorem 1.7 depends on τ as also illustrated by the cusp point coordinates.

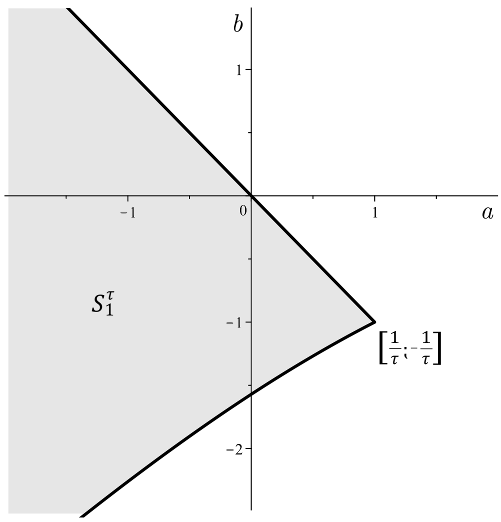

Figure 1.5: Stability region $\mathcal{S}_1^\tau$ for (1.5) depicted for the value $\tau = 1$.

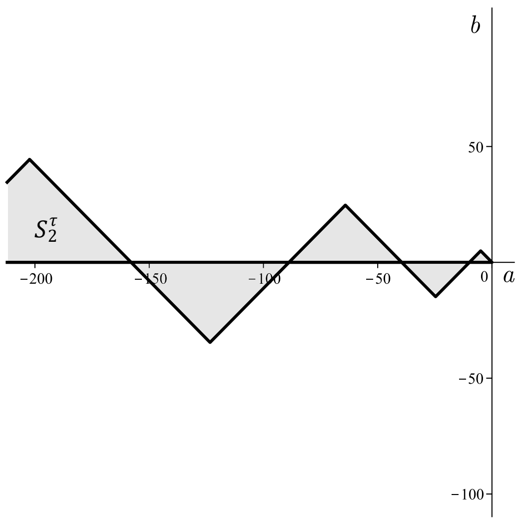

Figure 1.6: Stability region $\mathcal{S}_2^\tau$ for (1.6) depicted for the value $\tau = 1$.

Second-order differential equation with both delayed and undelayed terms

For better comparison, let us also consider the second-order system

$$y''(t) = ay(t) + by(t - \tau), \qquad t \in [0, \infty), \tag{1.6}$$

where a, b are real and $\tau > 0$ is a constant real lag. Its stability properties can be derived from [5] (although the case $b < 0$ was not explicitly discussed there) to obtain

Theorem 1.9. *Let $a, b \in \mathbb{R}$, $\tau \in \mathbb{R}^+$ and $\ell \in \mathbb{Z}_0^+$ be such that*

$$\ell^2 \frac{\pi^2}{\tau^2} < |a| < (\ell+1)^2 \frac{\pi^2}{\tau^2} .$$

Then (1.6) is asymptotically stable if and only if $a < 0$ and either

$$0 < b < \min(-\ell^2 \frac{\pi^2}{\tau^2} - a, (\ell+1)^2 \frac{\pi^2}{\tau^2} + a) \qquad \text{for } \ell \text{ being zero or even,}$$

or

$$0 > b > \max(\ell^2 \frac{\pi^2}{\tau^2} + a, -(\ell+1)^2 \frac{\pi^2}{\tau^2} - a) \qquad \text{for } \ell \text{ being odd.}$$

Corollary 1.10. *The stability region of (1.6) is given by*

$$\mathcal{S}_2^\tau = \bigcup_{j=0}^{\infty} \left(\left\{ (a,b) \in \mathbb{R}^2 : 0 < b < \min(-(2j)^2 \frac{\pi^2}{\tau^2} - a, (2j+1)^2 \frac{\pi^2}{\tau^2} + a) \right\} \right.$$
$$\left. \cup \left\{ (a,b) \in \mathbb{R}^2 : 0 > b > \max((2j+1)^2 \frac{\pi^2}{\tau^2} + a, -(2j+2)^2 \frac{\pi^2}{\tau^2} - a) \right\} \right) .$$

Comparing Theorems 1.7 and 1.9 we see very different stability conditions for the first and second-order equations. This difference is demonstrated in Figures 1.5 and 1.6 in the form of quite distinct shapes of the corresponding stability regions. The transition between them with continuous change of derivative order will be one of the interest of the following chapters.

1.3 Classical characteristic equation approach

The characteristic equation is central to the stability analysis of linear differential systems, including those with delays. The usual approach involves substituting an exponential function with argument st (where s is a complex parameter) as a candidate solution into the system. This substitution transforms the differential system into an algebraic equation with s as the variable, such as

$$\begin{aligned} \det(s\mathrm{I} - A\exp(-s\tau)) &= 0 \qquad \text{for } (1.3)\,, \\ s - a - b\exp(-s\tau) &= 0 \qquad \text{for } (1.5)\,, \\ s^2 - a - b\exp(-s\tau) &= 0 \qquad \text{for } (1.6)\,. \end{aligned}$$

Unlike the characteristic equations of ordinary differential equations, which are polynomial in s, these equations are transcendental, leading to more challenges as they typically have an infinite number of roots.

The system stability is then determined by the location of the characteristic roots in the complex plane due to the well-known behaviour of exponential functions:

- If all roots have negative real parts, the system is asymptotically stable.
- If any root has a positive real part, the system is unstable.
- If the rightmost root, i.e. the root with the largest real part, lies on the imaginary axis, there might be stability or instability based on root multiplicities.

If entry parameters of the system are specified, the position of characteristic roots with respect to imaginary axis can be usually analyzed numerically case by case. However, if we need to design a system or its control, or if there is a risk of parameter uncertainty, this approach is very random and impractical. Thus, the focal point of our effort is a reformulation of stability conditions from terms of characteristic roots into terms of entry parameters.

That is usually done via finding stability boundary in the space of entry parameters. In other words, we are looking for all combinations of entry parameters yielding rightmost roots with zero real part. Due to continuous dependence of characteristic roots on entry parameters, we arrive at a hypersurface in the parameter space where the system transitions from stable to unstable. This approach is often called D-partition method, D-decomposition method or boundary locus method (see, e.g. [24, 30, 39, 46, 47, 54]).

If we consider a system of non-integer order, there is one significant difference: the exponential functions do not longer solve the system. We need to find an alternative way to derive the characteristic equation. The well-established practice is to employ Laplace transform method (for definition we refer to [17]) which leads to the same results for all the classical problems and is successfully used for fractional differential systems as well. In particular, for (1.2) it yields the well-known formula

$$\det(s^{\alpha}\mathrm{I} - A) = 0$$

illustrating that characteristic equations belonging to fractional differential problems typically contain non-analytic functions.

Further, we have to ensure the connection between location of characteristic roots and stability properties of the system other than the exponential argument (as exponentials are no longer solutions, see, e.g. [46, 53]).

As this thesis deals with problems combining both fractional orders and delays, the main challenges addressed in the following chapters are:

- Investigating the properties of roots of characteristic equations that are transcendental and non-analytic.
- Identifying efficient descriptions of stability boundaries in various parameter spaces, clarifying the role of derivative order and delay in shaping the corresponding stability regions.
- Deriving asymptotic expansions of various special functions, often by using the inverse Laplace transform.

Chapter 2

Analysis of one-term fractional delay differential systems

This chapter focuses on the stability, oscillatory and related asymptotic properties derived in author's papers [6, 10, 15, 37] for one-term FDDS

$$\mathrm{D}_0^\alpha y(t) = Ay(t-\tau), \qquad t \in (0,\infty) \tag{2.1}$$

where A is a constant real $d \times d$ matrix and $\alpha, \tau > 0$ are real scalars. The associated initial conditions have typically the form

$$y(t) = \phi(t), \qquad t \in [-\tau, 0]\,, \tag{2.2}$$

$$\lim_{t\to 0^+} y^{(j)}(t) = \phi_j, \qquad j = 0,\dots,\lceil\alpha\rceil - 1 \tag{2.3}$$

where all components of d-vector function ϕ are absolutely Riemann integrable on $[-\tau, 0]$ and ϕ_j are constant real d-vectors.

The presence of fractional derivative creates room for discussions regarding the proper choice of its lower limit a (see (1.1)) which coincides with the "time origin" of the system. In particular, one might ask why not to put this limit to $a = -\tau$? Similar issues were discussed as the problem of so-called initialization in [44]. Although this discussion is quite interesting, no matter the result it does not significantly affect the qualitative study, because the change of the lower limit is analogous to adding a forcing term on the right-hand side of (2.1). Hence, we adopt the standard approach and consider the lower limit of fractional operators to be zero.

The study of qualitative properties of (2.1) was approached by many authors from different angles. One of the first attempts was [29] dealing with scalar version of (2.1) with real parameter employing the Lambert function to discuss asymptotic properties of solutions. Many authors in 2005-2012, e.g. [16,40,54], studied problems of vector nature, often involving more fractional derivative terms and more time delays. However, the stability criteria were almost exclusively limited on conditions for locations of characteristic roots in complex plane without an explicit link to the entry parameters of the corresponding problem. The difficult practical use and lack of efficiency of such results were often mentioned by authors themselves (see, e.g. [16, 40]). To our knowledge, the first explicit stability criterion was published for scalar version of (2.1) in 2011 by [39].

With this background, we started our work on [6] in 2014 and managed to derive explicit stability criterion and the asymptotics of bounded solutions for (2.1) of low orders (less than one). Three years later, in [10], we expanded our scope to (2.1) of higher orders (greater than one) for which we thoroughly analyzed oscillatory properties which, to our knowledge, were not discussed to that extent in the literature at the time (see, e.g. [3]). In 2020, we consolidated these results in [37], providing a comprehensive summary of the stability and asymptotic properties of (2.1) across all orders. Additionally, we extended our findings to systems involving another, so-called Riemann-Liouville, fractional derivative requiring a different type of initial conditions. It was only in 2023, in [15], when we added an easy-to-use graphical approach to estimate the asymptotic behaviour of unbounded solutions based on properties of Lambert function.

The key results from these four papers serve as the foundation for the following sections. As (2.1) transitions into (1.2) when $\tau \to 0$, and reduces to (1.3) as $\alpha \to 1$, this chapter focuses on comparing the properties of (2.1) with its limit counterparts. In Section 2.1 we establish the structure of solutions to (2.1) and the role of so-called generalized delay exponentials whose asymptotic properties are analyzed in Section 2.2. Section 2.3 describes the decomposition of complex plane naturally imposed by characteristic roots with zero real parts. Finally, Sections 2.4 and 2.5 are devoted to asymptotically stable and unstable systems, respectively, namely to the evolution of stability conditions, asymptotics and oscillatory properties with changes in the derivative order α.

2.1 Structure of solutions

Regarding solving of linear fractional equations, the Laplace transform is one of the most powerful tools. The problem (2.1)-(2.3) is no different. Applying Laplace transform (see [6, 10, 37]), we quickly notice the significance of the following notion:

Definition 2.1. Let $A \in \mathbb{R}^{d\times d}$, let I be the identity $d \times d$ matrix and let $\alpha, \tau \in \mathbb{R}^+$. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d\times d}$ given by

$$R(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}\right)(t) \tag{2.4}$$

is called the fundamental matrix solution of (2.1). We note that $\mathcal{L}^{-1}$ denotes the standard inverse Laplace transform, i.e. $\mathcal{L}^{-1}(F(s))(t) = (2\pi\,\mathrm{i})^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s)\mathrm{e}^{st}\mathrm{d}s$.

The inverse matrix occurring in (2.4) suggests the well-known characteristic equation associated to (2.1)

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0\,, \quad \text{i.e.} \quad \prod_{i=1}^{n}(s^\alpha - \lambda_i\exp\{-s\tau\})^{n_i} = 0 \tag{2.5}$$

where λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A and n_i are their algebraic multiplicities.

The concept of fundamental matrix solution (for integer orders see, e.g. [22]) yields the following solution representation depending on the initial conditions:

Theorem 2.2. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha, \tau \in \mathbb{R}^+$ and R be the fundamental matrix solution of (2.1). Then the solution y of (2.1)–(2.3) is given by*

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1} R(t)\phi_j + \int_{-\tau}^{0} R(t-\tau-u)A\phi(u)\mathrm{d}u\,.$$

Proof. The assertion follows directly from the evaluation of inverse Laplace transform of (2.1)–(2.3), for details see [6, 37]. □

To use these findings for qualitative analysis, we have to find more nuanced description of the solution. Applying the theory of Jordan canonical matrices on the fundamental matrix solution, we discover a key role of the function introduced by

Definition 2.3. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. The generalized delay exponential function (of Mittag-Leffler type) is introduced via

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty} \binom{m+j}{j} \frac{\lambda^j (t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)} h(t-(m+j)\tau)$$

where h is the Heaviside step function.

Remark 2.4. We note that special choices of G function parameters yield functions known to solve special cases of (2.1). Indeed,

- $G_{1,1}^{\lambda,0,0}(t)$ reduces to classical exponential $\exp\{\lambda t\}$ solving $y'(t) = \lambda y(t)$,
- $G_{\alpha,1}^{\lambda,0,0}(t)$ coincides with one-parameter Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ solving the scalar version of (1.2) (see, e.g. [53]),
- $G_{1,1}^{\lambda,\tau,0}(t)$ is the delay exponential solving the scalar version of (1.3) (see, e.g. [2]).

The Laplace transform of the generalized delay exponential function of Mittag-Leffler type is

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \frac{s^{\eta-\beta}\exp\{-ms\tau\}}{(s^\eta - \lambda\exp\{-s\tau\})^{m+1}}\,, \tag{2.6}$$

which allows us to detail the fundamental matrix solution as follows.

Lemma 2.5. *The fundamental matrix solution (2.4) can be expressed as* $R(t) = T^{-1}\mathcal{G}(t)T$*, where* T *is a regular matrix and* $\mathcal{G}$ *is a block diagonal matrix with upper-triangular blocks* B_j *given by*

$$B_j(t) = \begin{pmatrix} G^{\lambda_i,\tau,0}_{\alpha,\alpha}(t) & G^{\lambda_i,\tau,1}_{\alpha,\alpha}(t) & G^{\lambda_i,\tau,2}_{\alpha,\alpha}(t) & \cdots & G^{\lambda_i,\tau,r_j-1}_{\alpha,\alpha}(t) \\ 0 & G^{\lambda_i,\tau,0}_{\alpha,\alpha}(t) & G^{\lambda_i,\tau,1}_{\alpha,\alpha}(t) & \cdots & G^{\lambda_i,\tau,r_j-2}_{\alpha,\alpha}(t) \\ 0 & 0 & G^{\lambda_i,\tau,0}_{\alpha,\alpha}(t) & \cdots & G^{\lambda_i,\tau,r_j-3}_{\alpha,\alpha}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G^{\lambda_i,\tau,0}_{\alpha,\alpha}(t) \end{pmatrix},$$

where $j = 1, \dots, J$ $(J \in \mathbb{Z}^+)$*,* r_j *is the size of the corresponding Jordan block of* A*.*

Proof. See [6, 10]. □

Summarizing the above-stated results, we arrive at the crucial assertion describing the role of G functions, which serves as foundation for our next analysis.

Theorem 2.6. *Let* $R(t)$ *be the fundamental matrix solution of (2.1). Further, let* λ_i $(i = 1, \dots, n)$ *be distinct eigenvalues of* A *and let* p_i *be the largest dimension of the Jordan block corresponding to the eigenvalue* λ_i*. Then the nonzero elements of* $R(t)$ *are given by linear combinations of the generalized delay exponential functions*

$$G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t), \qquad m = 0, \dots, p_i - 1, \quad i = 1, \dots, n.$$

2.2 Asymptotics of generalized delay exponentials

As mentioned in Section 1.3, the known asymptotics of exponential functions is underlying most of the qualitative analysis of integer-order problems. Analogously, the asymptotic properties of the generalized delay exponential function of Mittag-Leffler type (Definition 2.3) prove to be crucial in qualitative analysis of (2.1).

First, let us introduce the real-part ordering for the roots of the denominator in (2.6) where, for the sake of simplicity, we set $\eta = \alpha$ (note the link to the characteristic equation (2.5)). Let s_j $(j = 1, 2, \dots)$ be the roots of

$$s^\alpha - \lambda \exp\{-s\tau\} = 0$$

with ordering $\Re(s_j) \geq \Re(s_{j+1})$, particularly s_1 is called the rightmost root. Then we can write the foundational asymptotic result as

Lemma 2.7. *Let* $\lambda \in \mathbb{C}$*,* $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$*,* $\beta, \tau \in \mathbb{R}^+$*,* $m \in \mathbb{Z}^+_0$ *and* s_j *be roots of (2.6) with real-part ordering.*
(i) If $\lambda = 0$*, then*

$$G^{0,\tau,m}_{\alpha,\beta}(t) = \frac{(t - m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha + \beta)} h(t - m\tau).$$

(ii) If $\lambda \neq 0$, then

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \sum_{j=1}^{\infty} \sum_{\ell=0}^{m\cdot k_j} a_{j\ell}(t - m\tau)^{\ell} \exp\{s_j(t - m\tau)\} + P_{\alpha,\beta}^{\lambda,\tau,m}(t) \,,$$

where k_j is a multiplicity of s_j, $a_{j\ell}$ are suitable nonzero complex constants ($\ell = 0, \dots, mk_j$, $j = 1, 2, \dots$) and the term $P_{\alpha,\beta}^{\lambda,\tau,m}$ has the algebraic asymptotic behaviour expressed via

$$\begin{aligned} P_{\alpha,\beta}^{\lambda,\tau,m}(t) =& \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} \\ &+ \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \qquad \textit{as } t \to \infty. \end{aligned}$$

Proof. For the proof in its complete form see [6], for additional supplementary assertions needed for higher orders see [10].

Its idea is built around evaluation of G for large t through the inverse Laplace transform

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(R,\frac{\pi}{2}+\delta)} \frac{s^{\alpha-\beta}\exp\{st - ms\tau\}}{(s^{\alpha} - \lambda \exp\{-s\tau\})^{m+1}} \mathrm{d}s \,.$$

The symbol $\gamma\,(R, \pi/2 + \delta)$ denotes the specific oriented piecewise smooth curve (see Figure 2.1) formed by three segments, i.e. $\gamma(\mu, \theta) = \gamma_1 + \gamma_2 + \gamma_3$ where $\mu > 0$, $\theta \in (0, \pi]$ and

$$\begin{aligned} \gamma_1 &= \{s \in \mathbb{C} : \ s = -u\exp\{-\mathrm{i}\theta\},\, u \in (-\infty, -\mu)\} \,, \\ \gamma_2 &= \{s \in \mathbb{C} : \ s = \zeta\exp\{-\mathrm{i}u\},\, u \in [-\pi-\theta, \pi+\theta]\} \,, \\ \gamma_3 &= \{s \in \mathbb{C} : \ s = u\exp\{-\mathrm{i}\theta\},\, u \in (\mu, \infty)\} \,. \end{aligned}$$

The proof, apart from its considerable technical difficulty, utilizes several properties of characteristic roots. In particular, there exists $\delta > 0$ such that all roots s_j of (2.5) satisfy $|\operatorname{Arg}(s_i)| \neq \pi/2 + \delta$ and, moreover, that there are only finitely many of them satisfying $|\operatorname{Arg}(s_i)| < \pi/2+\delta$. For detail calculations of relevant root properties, see [6]. □

Remark 2.8. Notice that Lemma 2.7 focuses on non-integer values of α. The cases of integer values are already covered by the classical theory and are known to have exponential asymptotics. From the technical standpoint, the difference lies in the fact that for non-integer α the Laplace transform of G contains non-analytic function, while for integer α only analytic functions occur.

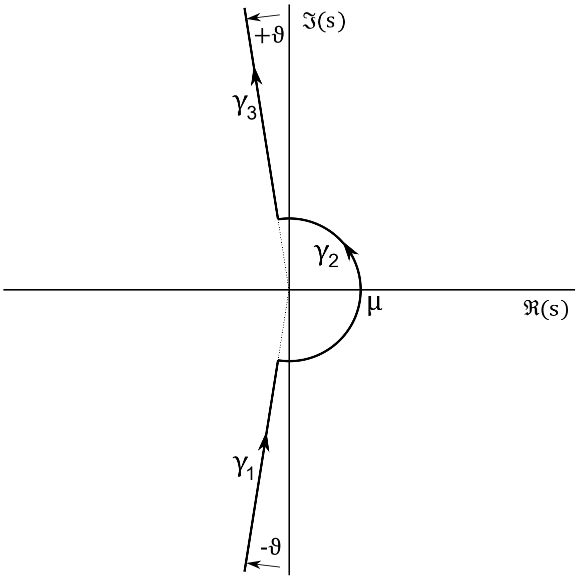

Figure 2.1: The curve $\gamma(\mu, \theta)$ used for evaluation of the inverse Laplace transform in the proof of Lemma 2.7.

2.3 Decomposition of eigenvalues' complex plane

Lemma 2.7 implies that, similarly to integer-order cases, the characteristic roots affect the stability properties primarily depending on the sign of their real parts. Hence, in this section we investigate the relation between locations of system matrix eigenvalues λ for (2.1) and zero real parts of characteristic roots of (2.5). In particular, we decompose the complex plane into regions such that eigenvalues chosen inside these regions guarantee nonzero real parts of the corresponding characteristic roots, and eigenvalues lying on boundaries of these regions imply at least one characteristic root with the zero real part.

Applying the standard approach of substituting $s = \mathrm{i}\,\varphi$ ($\varphi \in \mathbb{R}$) into factors of (2.5), i.e. $s^\alpha - \lambda \exp\{-s\tau\} = 0$, equating real and imaginary parts and rearranging with respect to $|\lambda|$ and $\mathrm{Arg}(\lambda)$. After a tedious calculations (see [10]), we can eliminate the parameter φ and define the regions as follows:

For any $\alpha > 0$ and $m \in \mathbb{Z}^+$ such that $0 < \alpha < 4m + 2$:

$$
\begin{aligned}
Q^\tau_\alpha(m) = \Big\{ &\lambda \in \mathbb{C} : \ |\lambda| < \Big(\frac{|\,\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^\alpha, \\
&\frac{\alpha\pi}{2} - 2m\pi < |\,\mathrm{Arg}(\lambda)| \le \frac{\alpha\pi}{2} - (2m-2)\pi \Big\} \\
\cup \Big\{ &\lambda \in \mathbb{C} : \ \Big(\frac{|\,\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + (2m-2)\pi}{\tau}\Big)^\alpha < |\lambda| < \Big(\frac{|\,\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^\alpha, \\
&|\,\mathrm{Arg}(\lambda)| > \frac{\alpha\pi}{2} - 2m\pi \Big\}
\end{aligned}
$$

where the sets $Q^\tau_\alpha(m)$ ($m \in \mathbb{Z}^+_0$) are defined to be empty whenever $\alpha \ge 4m + 2$.

Further, for $\alpha \in (0, 2)$ we add:

$$Q_\alpha^\tau(0) = \left\{ \lambda \in \mathbb{C} : |\lambda| < \left(\frac{|\operatorname{Arg}(\lambda)| - \alpha\pi/2}{\tau} \right)^\alpha, \; |\operatorname{Arg}(\lambda)| > \frac{\alpha\pi}{2} \right\}.$$

As illustrated by Figures 2.2-2.5, the sets $Q_\alpha^\tau(m)$ ($m \in \mathbb{Z}_0^+$) are disjoint and the infinite union of their closures covers the whole complex plane. In the next two section we detail the role of these sets in qualitative properties of (2.1).

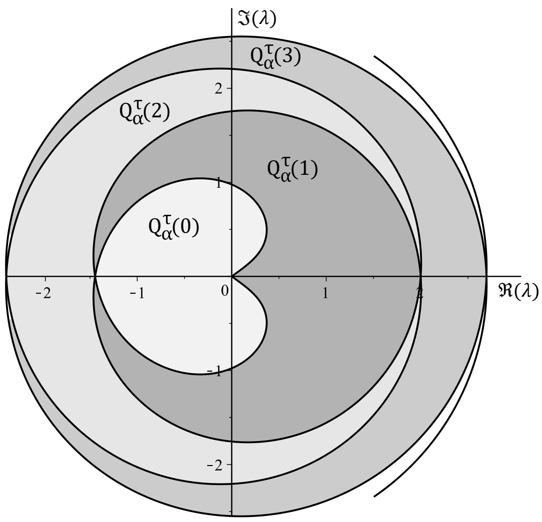

Figure 2.2: Decomposition of eigenvalues' complex plane for $\alpha = 0.4$, $\tau = 1$.

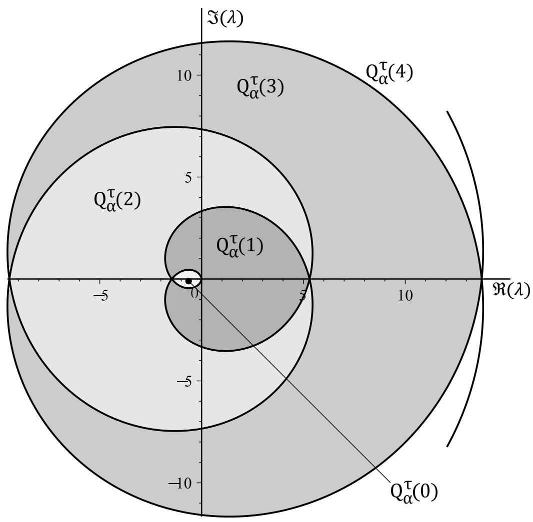

Figure 2.3: Decomposition of eigenvalues' complex plane for $\alpha = 1.1$, $\tau = 1$

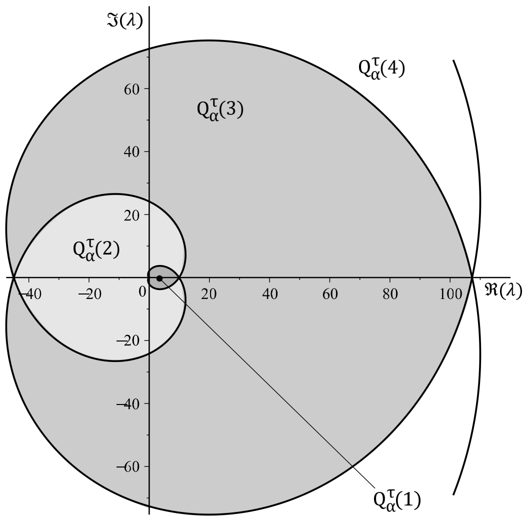

Figure 2.4: Decomposition of eigenvalues' complex plane for $\alpha = 2.1$, $\tau = 1$

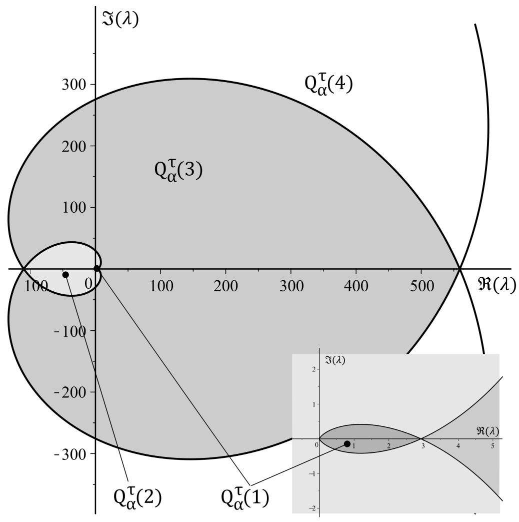

Figure 2.5: Decomposition of eigenvalues' complex plane for $\alpha = 3.1$, $\tau = 1$

2.4 Asymptotically stable systems

The calculations behind the complex plane decomposition from the previous section yield that the set $Q_\alpha^\tau(0)$ contains all the eigenvalues having solely characteristic roots with negative real parts. Thus, it coincides with the stability region for (2.1) which takes the form

$$\mathcal{S}_\alpha^\tau = \left\{ \lambda \in \mathbb{C} : \, |\lambda| < \left(\frac{|\operatorname{Arg}(\lambda)| - \alpha\pi/2}{\tau} \right)^\alpha , \, |\operatorname{Arg}(\lambda)| > \frac{\alpha\pi}{2} \right\} . \tag{2.7}$$

That enables us to write a fractional counterpart to Theorem 1.4 and simultaneously a delay counterpart to Theorem 1.1 as follows

Theorem 2.9. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and $\alpha \in (0,2)$. Then (2.1) is asymptotically stable if and only if all eigenvalues λ_i $(i = 1,\ldots,d)$ of A are nonzero and satisfy*

$$\tau|\lambda_i|^{1/\alpha} < |\operatorname{Arg}(\lambda_i)| - \alpha\pi/2 \, .$$

Moreover, if $\alpha \notin \mathbb{Z}^+$, then the convergence to zero is of algebraic type; more precisely, for any solution y of (2.1) there exists a suitable integer $j \in \{0,\ldots,\lceil\alpha\rceil\}$ such that $\|y(t)\| \sim t^{j-\alpha-1}$ as $t \to \infty$ (the symbol $\|\cdot\|$ means a norm in $\mathbb{R}^d$).

Proof. The proof is based on Theorems 2.2 and 2.6 combined with Lemma 2.7. Its main challenge lies in asymptotic evaluation of the integral term $\int_{-\tau}^0 R(t-\tau-u)A\phi(u)\mathrm{d}u$. For details see [6, 10, 37]. □

Figures 2.6 and 2.7 illustrate evolution of the stability region for increasing α. For all $\alpha \in (0,1) \cup (1,2)$ the stability boundary has a cusp point at the origin from which it continues symmetrically above and below real axis with tangents $\pm\alpha\pi/2$, respectively. For $\alpha = 1$, the cusp point smoothens as the tangents align with the imaginary axis.

Figures 2.8 and 2.9 show the shape of the stability region for α close to integer-order values one (compare to Figure 1.3) and two (the stability region vanishes). Figure 2.10 outlines the effect of decreasing τ causing expansion of the stability region up to the undelayed case for $\tau \to 0^+$ (compare to Figures 1.1 and 1.2).

The most puzzling insight brought by changing α in (2.1) is depicted in Figure 2.11 where we see shape of stability region for the values α close to zero. Although for all $\alpha > 0$ the positive reals lie outside of stability region, we see that the limit shape for $\alpha \to 0^+$ tends to a circle. As shown in Figure 1.4, the circle is the known stability region for difference equation (1.4) which, in a certain sense, can be seen as (2.1) with $\alpha = 0$. This remarkable connection suggests the potential of fractional-order derivatives to provide transition not just between integer-order differential systems but also between the differential and difference systems (for more comments see [6]).

The approach originating from Lemma 2.7 and Theorems 2.2 and 2.6 allows to address also the boundary of stability region.

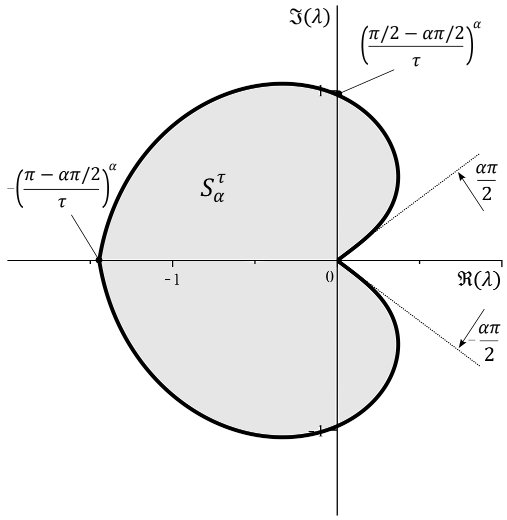

Figure 2.6: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.4$, $\tau = 1$.

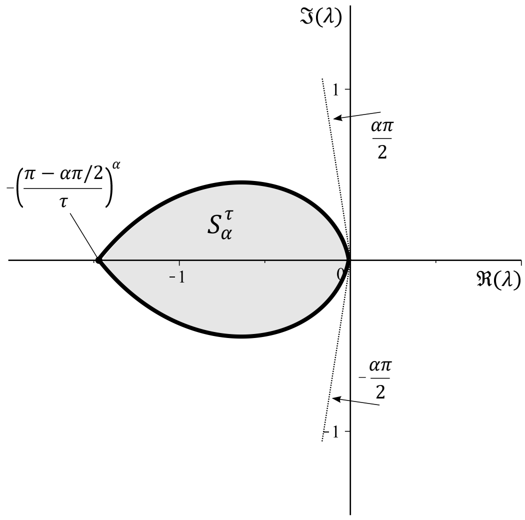

Figure 2.7: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 1.1$, $\tau = 1$.

Theorem 2.10. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and $\alpha \in (0,2)$. Then (2.1) is stable if and only if all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau$ or its boundary $\partial\mathcal{S}_\alpha^\tau$, and all the ones lying on the boundary have same algebraic and geometric multiplicities.*

Proof. See [6, 37]. □

Remark 2.11. (i) Comparing Theorems 2.9 and 2.10 we see that while presence of an eigenvalue on the stability boundary removes asymptotic stability, it preserves the stability provided it has the same same algebraic and geometric multiplicities. Consequently, for scalar version of (2.1), the system is stable if and only if all eigenvalues lie in the closure of $\mathcal{S}_\alpha^\tau$ (see also [39]).
(ii) Although this thesis deals with fractional derivatives of Caputo type, it is worth noting that (2.1) with a Riemann-Liouville derivative has nearly the same stability properties. The only difference occurs when the zero eigenvalue is present as proved in [37]. Specifically, if $\alpha < 1$ and the maximum size of any Jordan block associated with the zero eigenvalue is less than $1/\alpha$, the asymptotic stability appears.

Oscillatory properties of asymptotically stable systems

Theorem 1.6 implies that in the case of the first-order delay system (1.3), the oscillations occur for almost all λ (more precisely, some solutions of (1.3) do not oscillate, if some eigenvalue lies in $[-1/(\tau e), \infty)$, which is a set of zero measure). In the case of $\alpha \neq 1$, the situation is very different. The following assertion shows that there are no oscillatory solutions tending to zero.

Theorem 2.12. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let (2.1) be stable. If all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau \cup \{0\}$, then all nonzero solutions are non-oscillatory.*

Proof. The proof builds on Lemma 2.7 and the fact that in asymptotically stable case the non-oscillating algebraic term $P_{\alpha,\beta}^{\lambda,\tau,m}(t)$ dominates the oscillating exponential

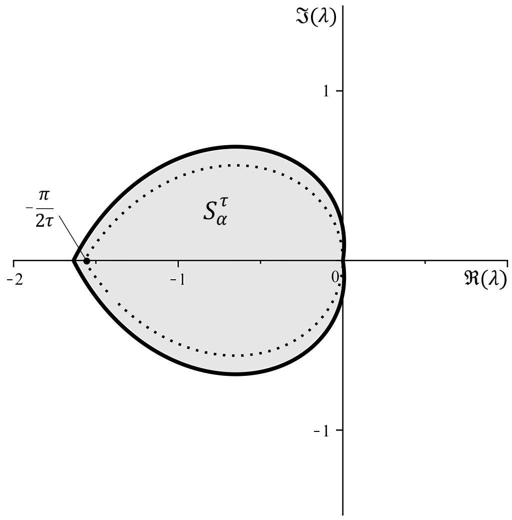

Figure 2.8: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.9$, $\tau = 1$ (the corresponding limit case $\alpha \to 1$ is dotted).

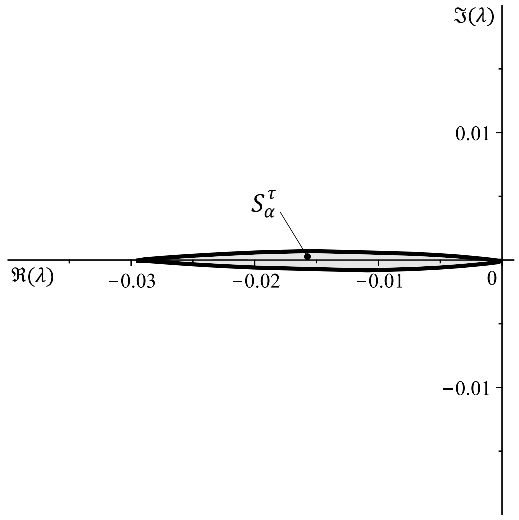

Figure 2.9: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 1.9$, $\tau = 1$ (the corresponding limit case $\alpha \to 2^-$ is an empty set).

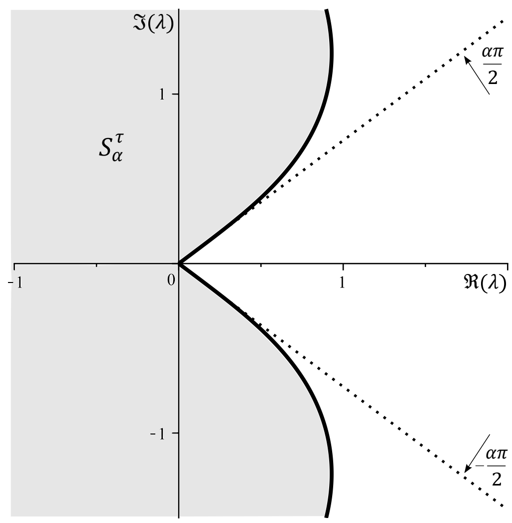

Figure 2.10: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.4$, $\tau = 0.1$ (the corresponding limit case $\tau \to 0^+$ is dotted).

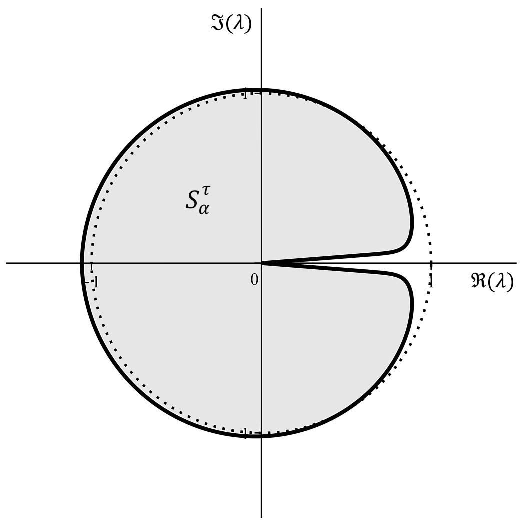

Figure 2.11: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.05$, $\tau = 1$ (the corresponding limit case $\alpha \to 0^+$ is dotted).

functions. In case of the zero eigenvalue, the additional term is also non-oscillatory no matter the multiplicity of the zero eigenvalue. □

2.5 Unstable systems

The techniques used in [6] to discuss properties of asymptotically stable systems turn out to be effective also in the case of unstable system. In particular, they enable us to describe the supremum asymptotics of the unbounded solutions as follows.

Theorem 2.13. *Let $A \in \mathbb{R}^{d\times d}$ and $\alpha, \tau \in \mathbb{R}^+$. Let λ_i be all distinct eigenvalues of A ($i = 1, \dots, n$) and let (2.1) is not stable. Then solutions $y(t)$ of (2.1) admit three types of asymptotics:*
(i) Let $\lambda_1 = 0$ be the zero eigenvalue of A with algebraic multiplicity greater than geometric one and let p_1 be the maximal size of Jordan blocks corresponding to λ_1. Further, let $\lambda_i \in \mathcal{S}_\alpha^\tau$ for all $i = 2, \dots, n$. Then

$$\|y(t)\| \sim t^{(p_1-1)\alpha} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

(ii) Let λ_i ($i = 1, \dots, \ell \le n$) be nonzero eigenvalues of A lying on $\partial\mathcal{S}_\alpha^\tau$ with algebraic multiplicity greater than geometric one and let p_i be the maximal size of Jordan blocks corresponding to λ_i ($i = 1, \dots, \ell$). Further, let $\lambda_i \in \mathcal{S}_\alpha^\tau$ for all $i = \ell + 1, \dots, n$ provided $\ell < n$ and $p = \max(p_1, \dots, p_\ell)$. Then

$$\|y(t)\| \sim_{sup} t^{p-1} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

(iii) Let λ_i ($i = 1, \dots, \ell \le n$) be eigenvalues of A located outside $\mathrm{cl}(\mathcal{S}_\alpha^\tau)$ and let s_1 be the rightmost root of (2.5). Further, let λ_j, $j \in L \subset \{1, \dots, \ell\}$ be eigenvalues of A such that (2.5) with $\lambda = \lambda_j$ has the zero s_1 and let p be the maximal size of Jordan blocks corresponding to λ_j, $j \in L$. Then

$$\|y(t)\| \sim_{sup} t^{p-1} \exp\{\Re(s_1)t\} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

Proof. The details see in [10]. □

To obtain an actually effective (and non-improvable) asymptotic result for the solutions of (2.1), we have to look at the problem inversely. More precisely, for a given complex $\lambda \notin \mathcal{S}_\alpha^\tau$, we need to find (nonnegative) real values u, v such that the rightmost root s_1 of (2.5) satisfies $\Re(s_1) = u$, $|\Im(s_1)| = v$.

This nontrivial question can be addressed using properties and methods of Lambert function, i.e. the function introduced as a solution of $W(z)\exp(W(z)) = z$, $z \in \mathbb{C}$ (see, e.g. [28]). In [15] we developed a framework allowing to evaluate the precise asymptotic envelop of the unbounded solutions and also the corresponding asymptotic frequency of oscillations. For the sake of simplicity, only scalar version of (2.1) with complex coefficient λ was considered. The findings can be summarized in the following

Theorem 2.14. *Let $\alpha \in (1, \infty)$, $\tau \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$. If $\lambda \notin \mathcal{S}_\alpha^\tau$, then, for any solution $y(t)$ of $\mathrm{D}_0^\alpha y(t) = \lambda y(t - \tau)$ it holds*

$$y(t) = \exp(ut)(c\exp(\mathrm{i}\,vt) + o(1)) \quad \text{as} \quad t \to \infty$$

where c is a complex constant, $u \geq 0$ is the unique solution of

$$\alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)} = |\mathrm{Arg}(\lambda)|\ ,$$

$v > 0$ is the unique solution of

$$\frac{v^\alpha}{\sin^\alpha\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)}\exp\big(\tau v\cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\big) = |\lambda|, \quad \textit{if } |\mathrm{Arg}(\lambda)| > 0\,,$$

and $v = 0$ if $\mathrm{Arg}(\lambda) = 0$.

Proof. The first and simple part is to express the characteristic roots s_k ($k \in \mathbb{Z}$) of (2.5) in terms of Lambert function, i.e.

$$s_k = \frac{\alpha}{\tau}W_k\left(\frac{\tau}{\alpha}\lambda^{1/\alpha}\right)\,, \qquad k \in \mathbb{Z}\,,$$

where W_k is the kth branch of the Lambert function. The key part is to prove the existence of ordering put on Lambert functions branches, namely that $\Im(W_k(z)) \leq \Im(W_{k+1}(z))$ for all $k \in \mathbb{Z}$ and $z \neq 0$ and $\Re(W_0(z)) \geq \Re(W_k(z))$ for all $k \in \mathbb{Z}$. The proof is then concluded by technically challenging calculations leading to the equations for u, v depending on α, τ, λ. For details we refer to [15]. □

Remark 2.15. (i) Note that Theorem 2.14 is not formulated for $\alpha < 1$. That is a consequence of $1/\alpha$ occurring in the argument of the Lambert function which for $\alpha < 1$ might introduce some additional roots which do not actually solve (2.5). This problem does not seem to be solvable in the framework of Lambert function method and, to the author's knowledge, remains open.
(ii) Figure 2.12 depicts a practical "map" allowing us to quickly estimate asymptotic modulus and oscillation frequency for the solution of $\mathrm{D}_0^\alpha y(t) = \lambda y(t-\tau)$ based on location of λ in the complex plane.
(iii) Theorem 2.14 considers only scalar version of (2.1) with complex coefficient. If we deal with the vector version of (2.1) and the system matrix A has eigenvalues with the same algebraic and geometric multiplicities, we just have to apply Theorem 2.14 for every eigenvalue and combine the results (Figure 2.12 also applies). In case of different algebraic and geometric multiplicities, the estimates for asymptotic frequencies are still valid and the estimates for the modulus have to adjusted by polynomial multiplication.

Oscillatory properties of unstable systems

As in the stable case, (2.1) (for $\alpha \neq 1$) does not have any combination of entry parameters guaranteeing oscillations of all solutions. On the other hand, there are combinations that ensure no oscillatory solutions.

Theorem 2.16. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let (2.1) be unstable. If all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau \cup \{0\} \cup (Q_1(\alpha,\tau) \cap \mathbb{R})$, then all nonzero solutions are non-oscillatory.*

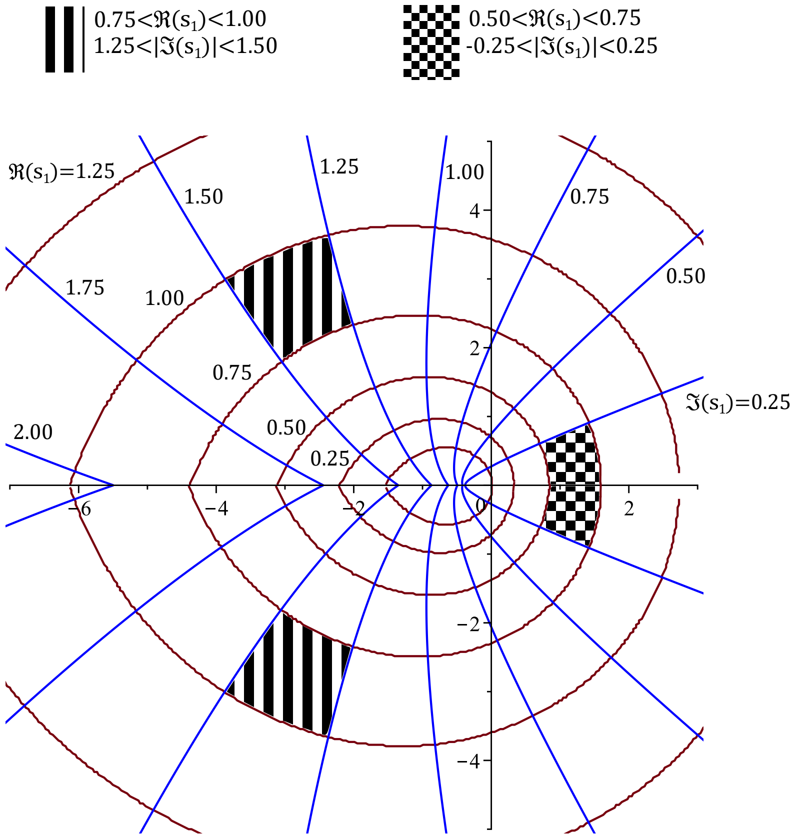

Figure 2.12: The blue curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_1 of (2.5) satisfies $\Im(s_1) = v$, and the particular orange curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_1 of (2.5) satisfies $|\Re(s_1)| = v$ (the scenario corresponds to $\alpha = 1.2$ and $\tau = 1$). As an example, there are highlighted curvilinear rectangles representing sets of all $\lambda \in \mathbb{C}$ yielding $\Re(s_1)$ and $\Im(s_1)$ from a certain range.

Proof. The outline of the prove is following, for details we refer to [10].

It can be seen from Lemma 2.7 that oscillatory solutions can occur only if there is a positive real characteristic root. Further, it is possible to prove that (2.5) has a positive real root if and only if λ is a positive real and this root is simple, unique and it is the rightmost root of (2.5).

Then we employ properties of $Q^{\tau}_{\alpha}(m)$ introduced in Section 2.3. In particular, that there exist just m ($m = 0, 1, \dots$) characteristic roots of (2.5) with a positive real part (while remaining roots have negative real parts) if and only if $\lambda \in Q^{\tau}_{\alpha}(m)$. Moreover, (2.5) has a root with the zero real part if $\lambda \in \partial[Q^{\tau}_{\alpha}(m)]$ for some $m = 0, 1, \dots$. □

Remark 2.17. Similarly to Theorems 2.12 and 2.16, oscillatory solutions can occur in some cases only for a particular choice of initial conditions (see [10]).

Chapter 3

Analysis of two-term fractional delay differential equations

This chapter summarizes the key findings related to two-term FDDE from author's papers [11–13]. Building on our analysis of one-term FDDS, it would seem natural to turn to

$$\mathrm{D}_0^\alpha y(t) = Ay(t) + By(t-\tau)\,,$$

where A, B are real $d \times d$ matrices and τ is a positive real time delay. Moreover, such a mathematical model would provide a large application potential (see, e.g. [40,43]), especially in control theory regarding stabilization of equilibria of fractional dynamical systems via delayed feedback controls. Although addition of $Ay(t)$ on the right-hand side looks quite straightforward, it highly increases the difficulty of the studied problem. Even the classical case $\alpha = 1$ is still generally unsolved (see, e.g. [4, 31, 46, 55]).

That is why we focus on the proper development of stability theory for the scalar case, namely two-term fractional delay differential equation (FDDE)

$$\mathrm{D}_0^\alpha y(t) = ay(t) + by(t-\tau)\,, \tag{3.1}$$

where a, b are real coefficients, $\tau > 0$ is a real lag and $\alpha \in (0, 2)$. Similarly as for (2.1), the associated initial conditions are considered as

$$y(t) = \phi(t)\,, \qquad t \in [-\tau, 0)\,, \tag{3.2}$$

$$\lim_{t\to 0^+} y^{(j)}(t) = \phi_j\,, \qquad j = 0, 1 \tag{3.3}$$

where ϕ is absolutely Riemann integrable on $[-\tau, 0)$ and ϕ_j are reals.

The topic of stability and asymptotic analysis of FDDEs attracts the attention of many authors. Before 2016, significant majority of corresponding stability results was derived as parametric equations or implicit relations for the stability boundary or usually as an outcome of the D-decomposition method and an appropriate root locus. For such or similar stability results on (3.1) (with $\alpha \in (0, 1)$) we refer to [1,30], but the trend is evident from the literature even for simpler cases (see, e.g. [16, 40, 41, 54]). Hence, we decided to focus on the formulation of explicit stability criteria for (3.1), as they provide much more accessible and practical tool in comparison to the usual parametric or implicit ones. In [13] we succeeded for the derivative order less than one and managed to find the explicit description of stability region in the (a, b)-plane and the formula for the change from stability to instability with respect to increasing τ which is present also in the first-order case.

It is well-known that the integer-order linear delay dynamical systems may change their stability into instability with growing time delay not just once, but repeatedly back and forth. This interesting phenomenon, often referred to as stability switching, is still a current subject of research as exemplified, e.g. by [19,42,47,51] where values of stability switches are described via parameters of the corresponding integer-order system. Thus, the occurrence of stability switching for FDDEs is a natural topic discussed in the second decade of 21st century, e.g. in [56, 57]. Our papers [11,12] are mostly devoted to this area, the former one considering (3.1) with imaginary coefficient a and α less than one, and the latter one with α between one and two. In both the cases we managed to derive explicit values of stability switches as well as conditions for so-called delay-independent stability. Moreover, [12] clarifies the continuous transition between qualitatively very different stability regions for (3.1) with $\alpha = 1$ and $\alpha = 2$.

The following sections are built on the main results of [11–13]. Section 3.1 elaborates on structure and asymptotics of solutions to (3.1). Then, unlike the Chapter 2, we focus less on precision of asymptotics and more on shape of stability regions and their dependence on system parameters. Section 3.2 deals with the case of derivative order less than one, Section 3.3 changes one of the system parameters from real to imaginary and Section 3.4 comes back to real (3.1) with order between one and two. Throughout the chapter, we put stress on the explicit stability conditions which are quite challenging for (3.1), among other things, because of the presence of stability switches (see Sections 3.3 and 3.4).

3.1 Structure and asymptotics of solutions

Because (3.1) shares with (2.1) its linearity, fractional derivative order as well as time delay, the structure of the solution is expected to be similar. The Laplace transform of (3.1)-(3.3) shows that the fundamental solution belongs to the family of functions

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^{\alpha} - a - b\exp[-s\tau]}\right)(t) \tag{3.4}$$

where $\alpha, \beta, \tau > 0$ and $a, b \in \mathbb{R}$. We can see that the generalized delay exponential function (with parameter $m = 0$) introduced by Definition 2.3 in the previous chapter is the special case of (3.4). Indeed, $\mathcal{R}^{0,b,\tau}_{\alpha,\beta}(t) = G^{b,\tau,0}_{\alpha,\beta}(t)$ (see (2.6)).

Using the inverse Laplace transform we arrive at the representation of the solution to (3.1)-(3.3) (compare to Theorem 2.2)

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil} \phi_j \mathcal{R}^{a,b,\tau}_{\alpha,j+1}(t) + b\int_{-\tau}^{0} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u\,. \tag{3.5}$$

The characteristic equation associated with (3.1) is implied by (3.4) and (3.5) in the expected form

$$s^\alpha - a - b\exp(-s\tau) = 0 \tag{3.6}$$

which has infinitely many complex roots (compare to (2.5) and to characteristic equations for the classical integer-order cases in Section 1.3).

The key auxiliary assertion, the counterpart to Lemma 2.7, deals with asymptotic properties of $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ functions.

Lemma 3.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1]$, $a, b \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$ and let s_i be roots of (3.6).*
(i) If $\Re(s_i) < 0$ for all s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim t^{\beta-\alpha-1} \text{ for } \alpha \neq \beta \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.$$

(ii) If there exists the zero root of (3.6) and $\Re(s_i) < 0$ otherwise, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \sim 1 \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{O}(t^{\beta-1}) \text{ for } \beta < 1 \qquad \text{as } t \to \infty\,.$$

(iii) If $\Re(s_i) \le 0$ for all s_i and some of the roots are purely imaginary, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim_{sup} 1 \qquad \text{as } t \to \infty\,.$$

(iv) If $\Re(s_i) > 0$ for some s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim_{sup} (Bt + C)\exp[Mt] \qquad \text{as } t \to \infty$$

where $M = \max_{s_i}(\Re(s_i))$ and reals $B, C \ge 0$ are such that $B + C > 0$.

Proof. The proof utilizes the technique already outlined in the proof of Lemma 2.7, with much higher technical difficulty, more branching to be considered with respect to the parameter values and with several adjustments (see [13, pages 344–349]).

The assumptions for the use of the technique needs the root analyses of (3.6), most importantly showing that for an arbitrary $0 < \omega < \pi$, the characteristic equation has no more than a finite number of roots s such that $|\operatorname{Arg}(s)| \le \omega$. □

Although Lemma 3.1 is formulated for real values of a, b and α less than one, it is only a technical matter to generalize it. The quality of the asymptotic estimates may be affected but the stability implications remains the same (as pointed out in [11, 12]).

Theorem 3.2. *Let $\alpha > 0$, $\tau > 0$ and a, b be complex numbers.*
(i) If all the roots of (3.6) have negative real parts, then (3.1) is asymptotically stable.
(ii) If there exists a root of (3.6) with a positive real part, then (3.1) is not stable.

Remark 3.3. Theorem 3.2 does not address the stability boundary. As we will discuss later, the stability boundary for (3.1) contains points of asymptotic stability, stability and also instability for various values of system parameters.

3.2 Stability regions for orders less than one

Let us consider (3.1) with $\alpha < 1$ and investigate the boundary locus for the corresponding (3.6), i.e. the set of all real couples (a, b) such that the characteristic equation admits a root with zero real part. Substituting $s = \pm\,\mathrm{i}\,\varphi$ into (3.6) and considering real and imaginary parts separately yields two qualitatively distinct parts of boundary locus: the line $a + b = 0$ (corresponding to the zero root) and the system of curves (corresponding to purely imaginary roots)

$$a_m(\rho) = \frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \qquad b_m(\rho) = -\frac{\rho^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \tag{3.7}$$

$m\pi < \rho < (m+1)\pi$, $m = 0, 1, \ldots$. The curves forming the boundary locus are depicted in the (a, b)-plane on Figure 3.1 (see also [30] where the authors redundantly considered multi-valued function s^α instead of the single-valued one).

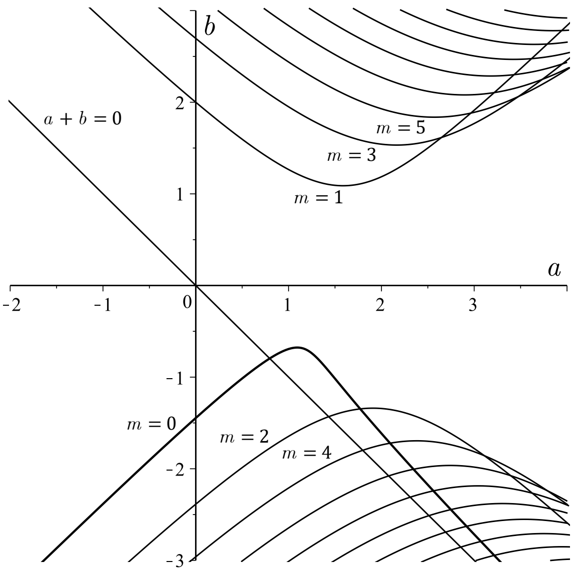

Figure 3.1: Boundary locus of (3.1) for $\alpha = 0.4$, $\tau = 1$.

The necessary link between the boundary locus curves and stability properties of (3.1) is provided by the following

Theorem 3.4. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then all roots of (3.6) have negative real parts if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve*

$$a = \frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \qquad \rho \in ((1-\alpha)\pi, \pi) \tag{3.8}$$

from below.

Proof. It follows from continuous dependence of roots of (3.6) on the coefficients a, b. This property particularly implies that the number of characteristic roots with a positive real part remains unchanged in all open sets whose boundaries are formed by the line $a+b=0$ or by some curves (3.7). Then it is enough to choose representatives of these open sets to specify the number of roots of (3.6) with positive real parts within these sets. For details see [13]. □

Remark 3.5. Theorem 3.4 implies that of all curves in the system (3.7) only a part of the curve characterized by $m=0$ affects the stability boundary (see Figure 3.2). Others lie in the region where (3.1) is not stable. Also, substituting $(1-\alpha)\pi$ into (3.8) enables us to calculate the coordinates of the cusp point (see also Figure 3.2).

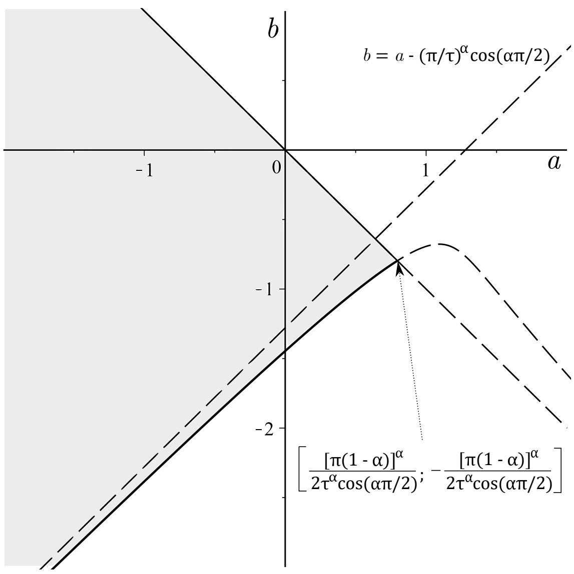

Figure 3.2: The stability region $\mathcal{S}_\alpha^\tau$ for (3.1) depicted for $\alpha = 0.4$ and $\tau = 1$.

The main objective of our effort is to derive explicit conditions determining the stability region. A related problem has been discussed in [1] where (3.6) is analysed for $a<0$. Here we present the assertion removing the restriction on a and yielding results in a simpler form due to the use of a different computational technique.

Theorem 3.6. *Let* $0<\alpha<1$, a, b *and* $\tau>0$ *be real numbers. Then* (3.1) *is asymptotically stable if and only if it holds either*

$$a \le b < -a \quad \textit{and} \quad \tau \textit{ is arbitrary}, \tag{3.9}$$

or

$$|a| + b < 0 \quad \textit{and} \quad \tau < \tau^* = \frac{(1-\alpha)\pi/2 + \arccos[(-a/b)\sin(\alpha\pi/2)]}{[a\cos(\alpha\pi/2) + (b^2 - a^2\sin^2(\alpha\pi/2))^{1/2}]^{1/\alpha}}\,. \tag{3.10}$$

Proof. The proof is based on rewriting the parametric equations (3.8) into the explicit ones via intersection analysis of the boundary locus curves, elimination of the parameter φ and careful operations with inverse trigonometric functions. For details see [13]. □

Remark 3.7. (i) Clearly, the stability region $\mathcal{S}_\alpha^\tau$ consists of pairs (a, b) such that either (3.9) or (3.10) holds. The condition (3.9) defines the region of delay-independent stability. The second condition, (3.10), shows dependence on the time delay, namely it indicates the one-time loss of stability with increasing τ reaching the value τ^*.
(ii) The delay-dependent part of stability region expands with decreasing time delay. If we consider the limit $\tau \to 0^+$, the stability region simplifies into half-plane $a + b < 0$. That agrees with the stability region for the scalar version of (1.2) with the coefficient $a + b$ which is the limit of (3.1) for $\tau \to 0^+$.
(iii) Considering (3.1) with $a = 0$, we obtain the scalar version of (2.1) with coefficient b which is asymptotically stable for $-(\pi/\tau - \alpha\pi/(2\tau))^\alpha < b < 0$ (see Figures 2.6 and 2.7). That corresponds to Theorem 3.6 as $-(\pi/\tau - \alpha\pi/(2\tau))^\alpha$ is the intersection between b-axis and the lower branch of the stability boundary.

The following theorem shows that for α less than one, the stability boundary fully corresponds to the situation when (3.1) is stable but not asymptotically stable.

Theorem 3.8. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then (3.1) is stable, but not asymptotically stable, if and only if either*

$$a + b = 0, \qquad a \leq \frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}, \tag{3.11}$$

or

$$|a| + b < 0, \qquad \tau = \tau^*, \quad \tau^* \text{ being the same expression as in (3.10).} \tag{3.12}$$

Proof. The assertion follows from application of Lemma 3.1, see [13]. □

Remark 3.9. The stability of (3.1) in the cusp point (the intersection of the line $a + b = 0$ and (3.8) is not analogue to the situation known from (1.5). It can be proved by a direct calculation that (1.5) at the cusp point (i.e. $a = -b = 1/\tau$) is not stable.

Due to the scalar nature of (3.1) and quite simple form of the stability boundary for of α less than one, we have quite comprehensive asymptotic description of solutions.

Lemma 3.10. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers and let y be a solution of (3.1).*
(i) Let (3.1) be asymptotically stable. Then

$$y(t) \sim t^{-\alpha} \qquad \text{or} \qquad y(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.$$

(ii) Let (3.1) be stable but not asymptotically stable. If (3.11) *is satisfied, then*

$$y(t) \sim 1 \qquad or \qquad y(t) = \mathcal{O}(t^{\alpha-1}) \qquad as\ t \to \infty\,.$$

If (3.12) holds, then

$$y(t) \sim_{sup} 1 \qquad or \qquad y(t) = \mathcal{O}(1) \qquad as\ t \to \infty\,.$$

(iii) Let (3.1) be unstable. Then $y(t) = \mathcal{O}(t\exp[Mt])$ *as* $t \to \infty$*, where* $M = \max_{s_i}(\Re(s_i))$*,* s_i *being roots of (3.6). Moreover, there exists a solution* y *of (3.1) such that*

$$y(t) \sim_{sup} t\exp[Mt] \qquad or \qquad y(t) \sim_{sup} \exp[Mt] \qquad as\ t \to \infty\,.$$

Proof. The proof is a consequence of Lemma 3.1 and (3.5). See [13] for details. □

Remark 3.11. As usual for asymptotically stable fractional dynamic systems, the decay rate of the solutions is algebraic, while in the classical case (1.5) it is known to be exponential.

3.3 Stability regions for orders less than one and an imaginary coefficient

Let us change the first coefficient of (3.1) into an imaginary one and study the problem

$$\mathrm{D}_0^\alpha y(t) = \mathrm{i}\,ay(t) + by(t-\tau)\,. \tag{3.13}$$

Although (3.13) might look artificially constructed, it actually plays a key role in the stability investigation of a planar fractional delay system

$$\begin{aligned} \mathrm{D}_0^\alpha x_1(t) &= ux_1(t-\tau) + vx_2(t) \\ \mathrm{D}_0^\alpha x_2(t) &= wx_1(t) + ux_2(t-\tau) \end{aligned}$$

with real entries u, v, w $(v, w \neq 0)$ which was the focus of [11]. We note that the corresponding classical case ($\alpha = 1$) was studied by [46] due to its stability switching nature.

Finding the boundary locus for the characteristic equation associated with (3.13),

$$s^\alpha - \mathrm{i}\,a - b\exp(s\tau) = 0\,, \tag{3.14}$$

starts in the same manner as in the previous section but soon key differences appear. First, (3.14) does not admit zero root. Second, purely imaginary roots induce the system of curves in a form

$$a_m(\rho) = \pm\frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \cos(\rho)}, \qquad b_m(\rho) = \frac{\rho^\alpha \cos(\alpha\pi/2)}{\tau^\alpha \cos(\rho)}\,, \tag{3.15}$$

$0 < \rho < \pi/2$ for $m = 0$ and $m\pi - \pi/2 < \rho < m\pi + \pi/2$ for $m \in \mathbb{Z}^+$ while $\rho + \alpha\pi/2 \neq m\pi$ for $m \in \mathbb{Z}_0^+$. Even though (3.15) looks formally similar to (3.7), the description of the stability boundary is now much more complicated as it is formed by parts of all curves (3.15) requiring calculations of infinitely many intersections. For the precise procedure we refer to [11] and state the end result:

Let us define two curves

$$\Gamma^+ = \bigcup_{m=0}^{\infty} \Gamma_{2m} \qquad \text{and} \qquad \Gamma^- = \bigcup_{m=0}^{\infty} \Gamma_{2m+1}$$

composed of the system of curves Γ_m in the (a, b)-plane given by (3.15) with the parameter restriction

$$\frac{(2m+1-\alpha)\pi}{2} - \theta^*_{m-2} < \rho < \frac{(2m+1-\alpha)\pi}{2} + \theta^*_m, \qquad m \in \mathbb{Z}_0^+$$

where $\theta^*_m \in (0, \alpha\pi/2)$ is the unique root of

$$-\frac{\sin(\theta + \alpha\pi/2)}{\sin(\theta - \alpha\pi/2)} = \left(\frac{(2m+3-\alpha)\pi}{\theta + (m + 1/2 - \alpha/2)\pi} - 1 \right)^{\alpha}, \quad m \in \mathbb{Z}_0^+ \tag{3.16}$$

and $\theta^*_{-2} = (\alpha - 1)\pi/2$, $\theta^*_{-1} = \pi/2$. Note that in the first relation of (3.15), both the sign cases have to be considered, and thus any curve Γ_m has two branches symmetric with respect to b-axis.

Using this notation we can write

Lemma 3.12. *Let $0 < \alpha < 1$, $\tau > 0$, $a \neq 0$ and b be real numbers. Then (3.13) is asymptotically stable if and only if the couple (a, b) is located inside the area bounded by Γ^+ from above and by Γ^- from below.*

Proof. The proof is using the connection between the growth of $|b|$ and number of characteristic roots with a positive real part, and analysis of intersections among Γ_m. Moreover, it employs appropriate asymptotic properties of the functions $\mathcal{R}^{\pm \mathrm{i}\, a,b,\tau}_{\alpha,\alpha}$, $\mathcal{R}^{\pm \mathrm{i}\, a,b,\tau}_{\alpha,1}$ which, as a by-product, also implies the algebraic decay rate of solutions. For details see [11]. □

Remark 3.13. (i) Lemma 3.12 describes parametrically the stability region $\mathcal{S}^\tau_\alpha$ of (3.13). Figure 3.3 depicts the known result for $\alpha = 1$ (see [47]) and Figures 3.4, 3.5 and 3.6 show the evolution of the stability region for decreasing derivative order. They illustrate that region of delay-independent stability occurs for any $\alpha < 1$.
(ii) If we consider lines connecting the origin and cusp points of Γ^+, we can prove that they have decreasing tangents with respect to increasing index of a given cusp point starting closest to b-axis (a similar comment is true also for Γ^-). This fact plays a central role in the context of stability switching.
(iii) Γ^+ has the tangent $\pm\cot(\alpha\pi/2)$ at the origin. Let us consider a line connecting

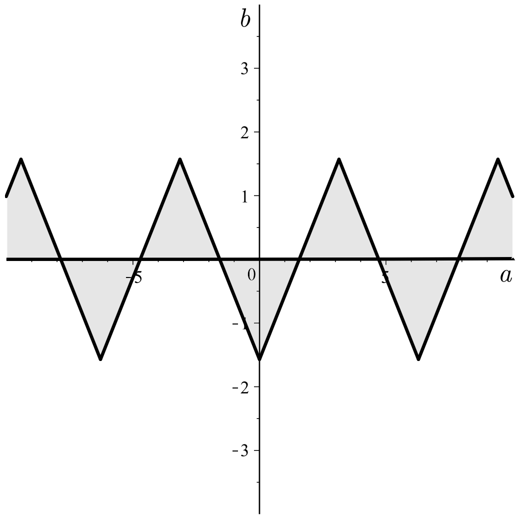

Figure 3.3: A classical result for the stability region $\mathcal{S}_1^\tau$ with $\tau = 1$.

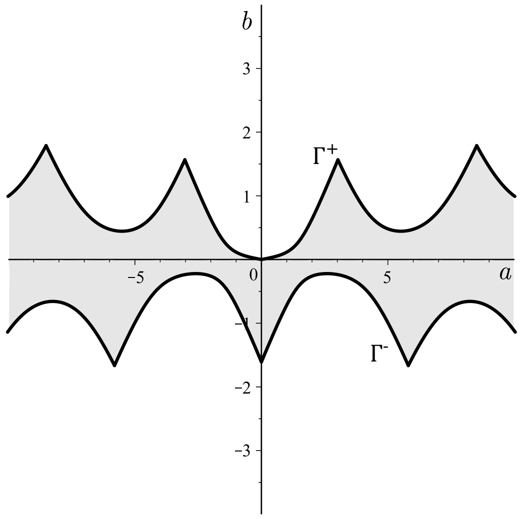

Figure 3.4: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.95$, $\tau = 1$.

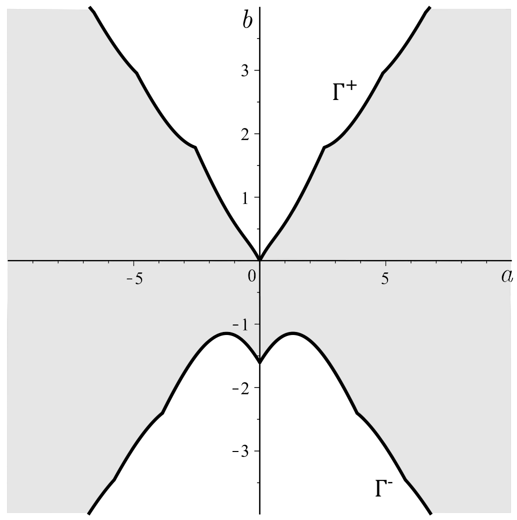

Figure 3.5: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.6$, $\tau = 1$.

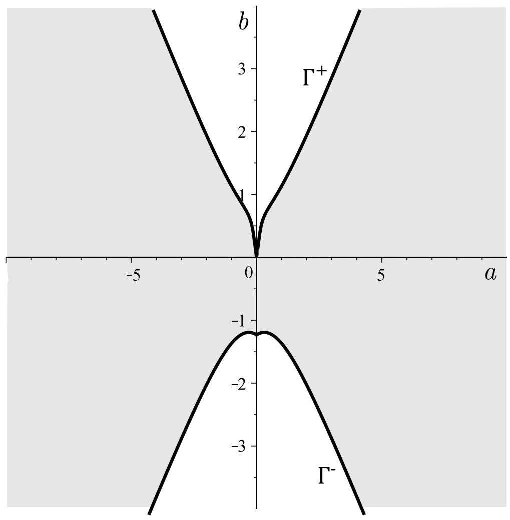

Figure 3.6: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.2$, $\tau = 1$.

origin with the first cusp point of Γ^+ (i.e. the endpoint of Γ_0). It can be calculated that the tangent of this line is equal to $\pm\cot(\alpha\pi/2)$ if and only if $\alpha = \alpha^*$ where

$$\alpha^* \approx 0.6150768144\,. \tag{3.17}$$

If $\alpha > \alpha^*$, then for some values of b/a the first stability switch changes instability into stability. If $\alpha < \alpha^*$, the first stability switch is always from stability to instability. For the full calculation we refer to [11].

The shape of the stability boundary for (3.13) is quite complex and its proper reformulation into the explicit form of stability conditions brings many challenges.

It seems that the most useful perspective is provided by considering the ratio b/a as presented in the main result of this section:

Theorem 3.14. *Let $0 < \alpha < 1$, $\tau > 0$ and a, b be real numbers, let α^* be given by (3.17) and let $\theta_m^* \in (0, \alpha\pi/2)$ be the unique root of (3.16) for $m \in \mathbb{Z}_0^+$. Further, assuming $|b|/|a| \geq \cos(\alpha\pi/2)$, let $n \geq 0$ be an even integer (if $b \geq 0$), or an odd integer (if $b < 0$), uniquely determined by*

$$\frac{\cos(\alpha\pi/2)}{\cos(\theta_n^*)} \leq \frac{|b|}{|a|} < \frac{\cos(\alpha\pi/2)}{\cos(\theta_{n-2}^*)}\,, \quad n \geq 2\,, \qquad \text{or} \qquad \frac{\cos(\alpha\pi/2)}{\cos(\theta_n^*)} \leq \frac{|b|}{|a|}\,, \quad n \in \{0, 1\}\,. \tag{3.18}$$

The zero solution to (3.13) is asymptotically stable if and only if any of the following conditions holds:

$$\frac{|b|}{|a|} < \cos(\alpha\pi/2)\,; \tag{3.19}$$

$$\cos(\alpha\pi/2) \leq \frac{b}{|a|} < \cot(\alpha\pi/2) \quad \text{and} \quad \tau \in \bigcup_{j=-1}^{n/2-1} (\tau_{2j,-1}^*, \tau_{2j+2,1}^*)\,; \tag{3.20}$$

$$\alpha > \alpha^*, \quad \cot(\alpha\pi/2) \leq \frac{b}{|a|} < \frac{\cos(\alpha\pi/2)}{\cos(\theta_0^*)} \quad \text{and} \quad \tau \in \bigcup_{j=0}^{n/2-1} (\tau_{2j,-1}^*, \tau_{2j+2,1}^*)\,; \tag{3.21}$$

$$\frac{b}{|a|} \leq -\cos(\alpha\pi/2) \quad \text{and} \quad \tau \in \bigcup_{j=-1}^{(n-1)/2-1} (\tau_{2j+1,-1}^*, \tau_{2j+3,1}^*) \tag{3.22}$$

where $\tau_{i,\kappa}^* = 0$ for negative integers i and $\tau_{i,\kappa}^* = \tau_{i,\kappa}^*(a, b)$ where $i \in \mathbb{Z}_0^+$, $\kappa = \pm 1$ and

$$\tau_{i,\kappa}^*(a, b) = \frac{(i + (1-\alpha)/2)\,\pi - \kappa \arccos\left(|a/b| \cos(\alpha\pi/2)\right)}{\left(\kappa\sqrt{b^2 - a^2\cos^2(\alpha\pi/2)} + |a|\sin(\alpha\pi/2)\right)^{1/\alpha}}\,.$$

Proof. The proof of this theorem requires many preliminary assertions such as guaranteeing existence, uniqueness and ordering of θ_m^* $(m \in \mathbb{Z}_0^+)$, derivation of α^* and analysis of the ratio b/a for points belonging to Γ^+ and Γ^-. In particular, for a given ratio $|b/a|$, an analytical description of delays τ such that $(a, b) \in \Gamma^+ \cup \Gamma^-$ has to be given. For details see [11, pages 7-13]. □

Remark 3.15. (i) We note that the nonlinear inequalities (3.18) enable us to determine the exact number of stability switches. For (3.20) then there are $n+1$ switches (odd number), for (3.21) there are n switches (even number) and for (3.22) there are n switches (odd number).
(ii) The condition (3.19) describes the region of delay-independent stability symmetric with respect to both axes. Decreasing α expands this region towards the cone $b = |a|$ which is the limit case for $\alpha \to 0^+$.
(iii) For $b/|a| > 0$ we can see in Figures 3.8, 3.9 and 3.10 the role of the value $\alpha = \alpha^*$. It separates two qualitatively different patterns of stability switching, namely for $\alpha \leq \alpha^*$ it always starts with the first switch changing stability into instability.

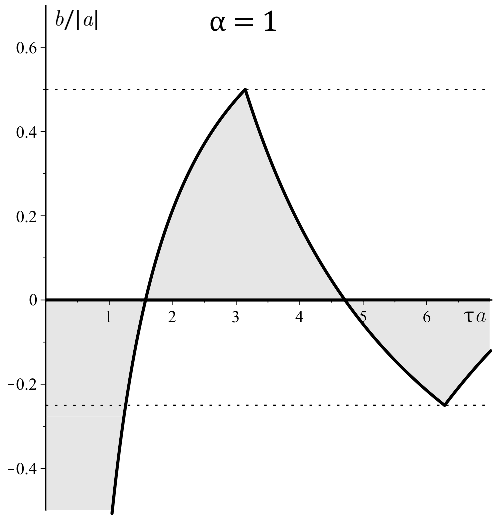

Figure 3.7: A classical result for the stability region $\mathcal{S}_1^\tau$ ($\alpha = 1$) in the $(\tau a, b/|a|)$-plane, see [46].

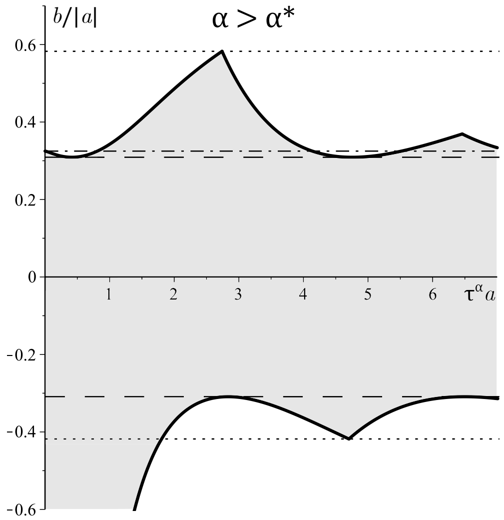

Figure 3.8: The stability region $\mathcal{S}_\alpha^\tau$ in the $(\tau^\alpha a, b/|a|)$-plane for $\alpha = 0.8$ (i.e. $\alpha > \alpha^*$).

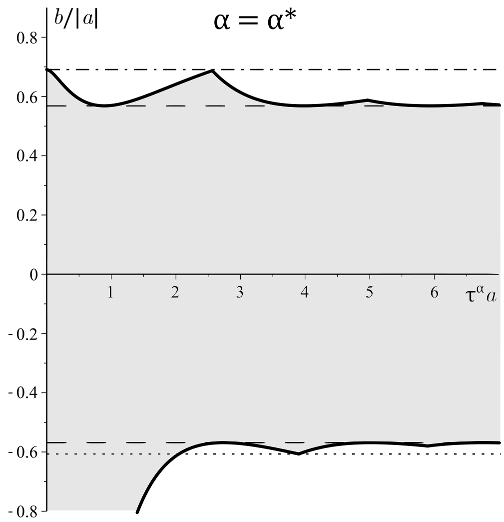

Figure 3.9: The stability region $\mathcal{S}_\alpha^\tau$ in the $(\tau^\alpha a, b/|a|)$-plane for $\alpha = \alpha^*$.

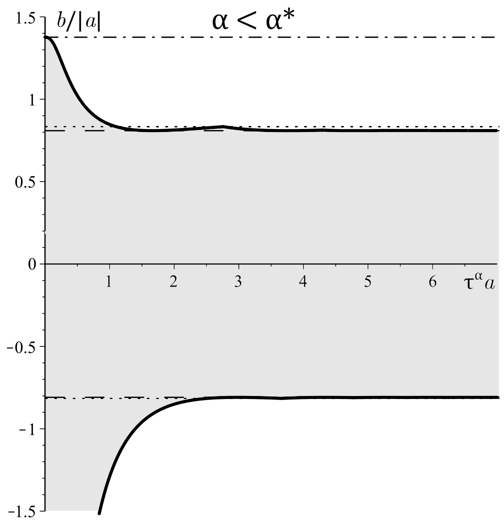

Figure 3.10: The stability region $\mathcal{S}_\alpha^\tau$ in $(\tau^\alpha a, b/|a|)$-plane for $\alpha = 0.4 < \alpha^*$.

3.4 Stability regions for orders from one to two

Let us consider (3.1) with $\alpha \in (1, 2)$. The boundary locus formulas have the same form as for the case $\alpha \in (0, 1)$, i.e. it is formed by the line $a + b = 0$ and the system of curves Γ_m in the (a, b)-plane given by (3.7). Properties of these curves for higher α significantly differ from the case $\alpha \in (0, 1)$ and are actually qualitatively more similar to (3.15) discussed in the previous section.

In [12] it proved to be useful for lucidity, to think of this system of curves from two perspectives: their asymptotes and intersections.

Lemma 3.16. *Let $\alpha \in (1,2)$, $\tau \in \mathbb{R}^+$ and let $\Gamma_m = \{(a,b) \in \mathbb{R}^2 : a = a_m(\rho),\ b = b_m(\rho),\ \rho \in (m\pi, m\pi+\pi)\}$ $(m = 0,1\dots)$ be the curves defined by (3.7). Then it holds:*
(i) The line $a+b=0$ is tangent to the curve Γ_0 at the origin, and the line

$$p_0^- : b = a - \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$$

is the asymptote to Γ_0 as $\rho \to \pi^-$. Moreover, $b_0(\rho) < a_0(\rho) - (\pi/\tau)^\alpha \cos(\alpha\pi/2)$, $b_0(\rho) < 0$ and $b_0(\rho) < -a_0(\rho)$ for all $\rho \in (0,\pi)$.
(ii) If m is a positive odd integer, then Γ_m has asymptotes p_m^+ (as $\rho \to m\pi^+$) and p_m^- (as $\rho \to (m+1)\pi^-$) given by

$$p_m^+ : b = a - \left(\frac{m\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_m^- : b = -a + \left(\frac{m\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_m(\rho) > 0$, $b_m(\rho) > a_m(\rho) - (m\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_m(\rho) > -a_m(\rho) + ((m\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $\rho \in (m\pi, (m+1)\pi)$.
(iii) If m is a positive even integer, then Γ_m has asymptotes p_m^+ (as $\rho \to m\pi^+$) and p_m^- (as $\rho \to (m+1)\pi^-$) given by

$$p_m^+ : b = -a + \left(\frac{m\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_m^- : b = a - \left(\frac{m\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_m(\rho) < 0$, $b_m(\rho) < -a_m(\rho) + (m\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_m(\rho) < a_m(\rho) - ((m\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $\rho \in (m\pi, (m+1)\pi)$.

Proof. The proof is of a technical nature utilizing limits calculations for (3.7). For details see [12]. □

Lemma 3.17. *Let $\alpha \in (1,2)$, $\tau \in \mathbb{R}^+$ and let et $\Gamma_m = \{(a,b) \in \mathbb{R}^2 : a = a_m(\rho),\ b = b_m(\rho),\ \rho \in (m\pi, m\pi+\pi)\}$ $(m = 0,1\dots)$ be the curves defined by (3.7). Further, let $X_{m,n} = (a_{m,n}, b_{m,n})$ be intersections of Γ_m and Γ_n (if they exist). Then it holds:*
(i) The intersection $(a_{m,n}, b_{m,n})$ exists (and it is unique) if and only if m,n have the same parity.
(ii) $a_{m,m+2k} < 0$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iii) $a_{m,m+2k} > a_{m,m+2(k+1)}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iv) $a_{m,m+2k} > a_{m+2\ell,m+2k+2\ell}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$ and $\ell = 1,2\dots$.

Proof. The question of analysing intersections of Γ_m, Γ_n is transformed into root study of an equation involving transcendental expression similar to (3.16). For detail see [12]. □

Remark 3.18. (i) Lemma 3.16 says that each curve Γ_m $(m = 0,1\dots)$ is contained in an infinite trapezoid consisting of its asymptotes and the a-axis. Each pair Γ_m, Γ_{m+1} shares a common asymptote as depicted in Figure 3.11.

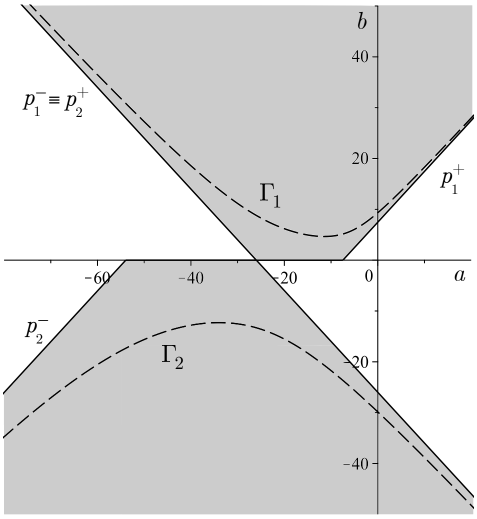

Figure 3.11: The common asymptote to Γ_1 and Γ_2 and the corresponding trapezoids for $\alpha = 1.8$ and $\tau = 1$.

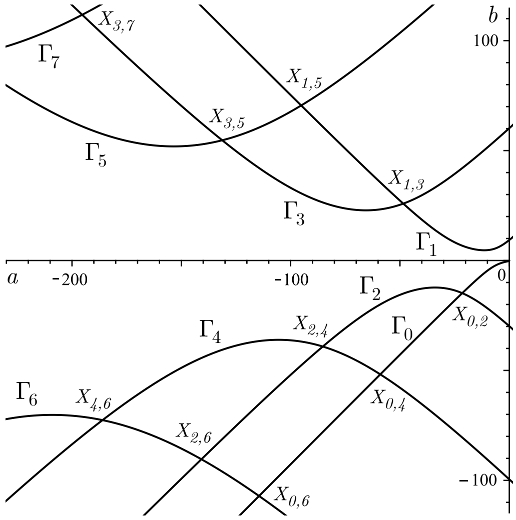

Figure 3.12: Some intersections $X_{m,n}$ for $\alpha = 1.8$, $\tau = 1$ and $m, n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(ii) Figure 3.12 demonstrates the locations and ordering of intersections $X_{m,n}$ described in Lemma 3.17.
(iii) A similar asymptotes and intersections analyses might be useful also in the case of (3.15), however it was not the point of study in [11].

In order to describe the stability region, we introduce the following notation. Let P be the line segment

$$a = -\rho, \quad b = \rho, \qquad \rho \in \left(0, \frac{(3\pi - \alpha\pi)^\alpha}{2\tau^\alpha |\cos(\alpha\pi/2)|}\right)$$

and let $\tilde{\Gamma}_m$ $(m = 0, 1, \dots)$ be the parts of Γ_m with the endpoints $X_{m,m-2}$ and $X_{m,m+2}$ given by its intersections with the neighbouring curves Γ_{m-2}, Γ_{m+2} (see Figure 3.12), by origin for $m = 0$ and by the second endpoint of P for $m = 1$. Further, we put

$$\Gamma^{AS} = \bigcup_{m=0}^{\infty} \tilde{\Gamma}_m \cup P \,.$$

Using this notation, the geometric description of the stability region is provided by the following assertion (compare to Theorem 3.4 and Lemma 3.12).

Theorem 3.19. *Let $\alpha \in (1, 2)$, $\tau \in \mathbb{R}^+$ and $a, b \in \mathbb{R}$. Then (3.1) is asymptotically stable, if the couple (a, b) is located inside the area containing the negative part of a-axis and bounded by Γ^{AS}.*
Moreover, (3.1) is not stable, if (a, b) lies inside the area containing the positive part of a-axis and bounded by Γ^{AS}.

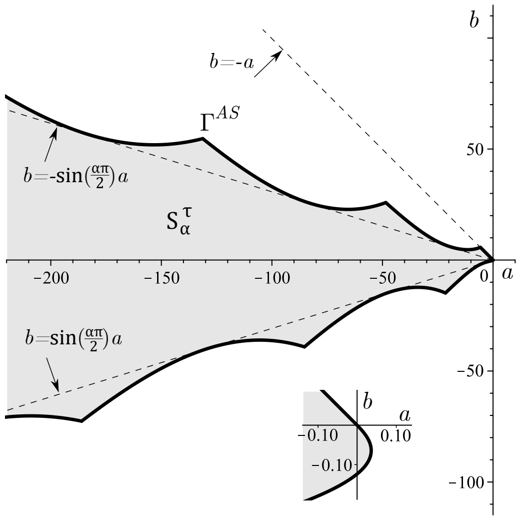

Figure 3.13: Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha^\tau$ of (3.1) for $\alpha = 1.8$ and $\tau = 1$.

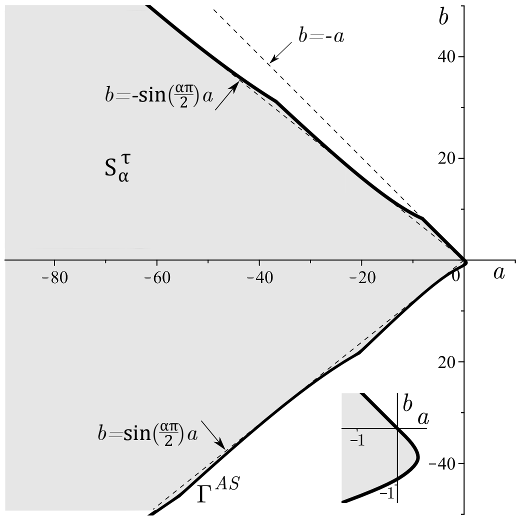

Figure 3.14: Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha^\tau$ of (3.1) for $\alpha = 1.4$ and $\tau = 1$.

Proof. In order to complete this proof, the preconditions similar to the case of $\alpha \in (0, 1)$ have to validated and recalculated (see proofs of Theorem 3.4 and Lemma 2.7). For details we refer to [12] and highlight here only the most interesting partial results such as:

If $(a, b) \in \Gamma_m$ (for unique m) and $|b|$ increases, then a new characteristic root with a positive real part appears.

If $a < 0$ and $b \in \mathbb{R}$, then there exists $\delta = \delta(\alpha, a) > 0$ such that all characteristic roots have negative real parts whenever $|b| < \delta$.

The nonzero characteristic roots depend on a, b continuously.

The case $\alpha \in (1, 2)$ stands out mainly due to the need to consider the occurrence of multiple roots. We proved that a characteristic root has multiplicity greater than one if and only if either it is zero or there exists an integer k such that $\alpha\rho - \rho + \tau r \sin(\rho) = k\pi$ and $\tau r \sin(\alpha\rho) + \alpha \sin(\alpha\rho - \rho) = 0$. Moreover, any characteristic root has multiplicity at most three. □

Remark 3.20. (i) Although Theorem 3.19 gives only sufficient conditions for asymptotic stability, its second part makes them near-optimal. The only additional points where the stability might occur, lie on the stability boundary given by Γ^{AS}.
(ii) The stability region $\mathcal{S}_\alpha^\tau$ described in Theorem 3.19 is depicted in Figures 3.13 and 3.14 including a detail of the situation near the origin. Comparing these details to Figure 1.5 suggests the limit transition for $\alpha \to 1^+$, as the rightmost point of Γ_1 changes into the cusp point appearing for $\alpha \leq 1$.

Our main goal is to obtain the explicit stability conditions, not just a geometric description of the stability boundary as in Theorem 3.19. In the sequel, we provide an alternative stability criterion for the case $a < 0$ that better agrees with the form of the conditions of Theorems 1.7 and 1.9. We do not consider the case $a > 0$ because the corresponding stability conditions are quite straightforward (see Figures 3.13 and 3.14).

Theorem 3.21. *Let $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b be real numbers and τ_ℓ^+, τ_ℓ^- be defined as*

$$\tau_\ell^{\pm} = \frac{(\ell + \frac{1\mp 1}{2})\pi + \frac{(2-\alpha)\pi}{2} \pm \arcsin\left(\left|\frac{a}{b}\right| \sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) \pm \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}}, \qquad \ell \in \mathbb{Z}_0^+ .$$

(i) If $-\sin(\alpha\pi/2) < b/a < \sin(\alpha\pi/2)$, then (3.1) is asymptotically stable.
(ii) If $b/a > \sin(\alpha\pi/2)$, then there exists an integer $N_1 \geq 0$ such that (3.1) is asymptotically stable for any $\tau \in (\tau_{2k-2}^-, \tau_{2k}^+)$, and it is not stable for any $\tau \in (\tau_{2k}^+, \tau_{2k+2}^-)$ where $k = 0,\dots,N_1$ (here we set $\tau_{-2}^- = 0$, $\tau_{2N_1+2}^- = \infty$).
(iii) If $-1 < b/a < -\sin(\alpha\pi/2)$, then there exists an integer $N_2 \geq 0$ such that (3.1) is asymptotically stable for any $\tau \in (\tau_{2k-1}^-, \tau_{2k+1}^+)$, and it is not stable for any $\tau \in (\tau_{2k+1}^+, \tau_{2k+3}^-)$ where $k = 0,\dots,N_2$ (here we set $\tau_{-1}^- = 0$, $\tau_{2N_2+3}^- = \infty$).
(iv) If $b/a < -1$, then (3.1) is not stable.

Proof. See [12]. □

Remark 3.22. (i) Comparing to Theorems 3.21 and 3.14 we notice two obvious differences: In the real case (Theorem 3.21), there is no special value α^* changing the switching pattern, and the real case is not symmetric with respect to b-axis (see also Figures 3.13 and 3.14). Theorem 3.21 is formulated without the exact condition for calculation for the number of stability switches, however these conditions (in form

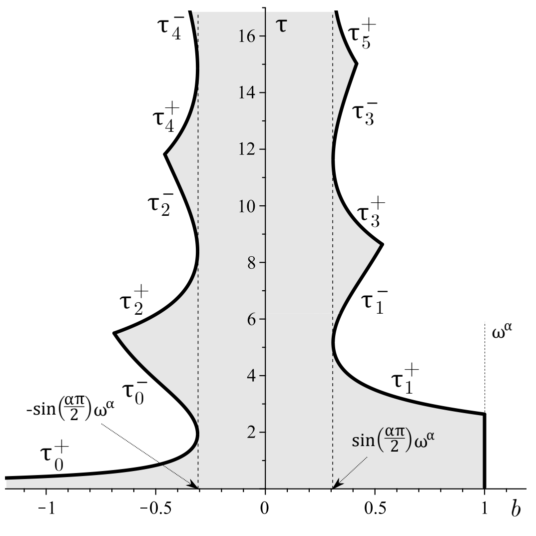

Figure 3.15: The stability region in (b,τ)-plane for $\alpha = 1.8$ and $a = -1$.

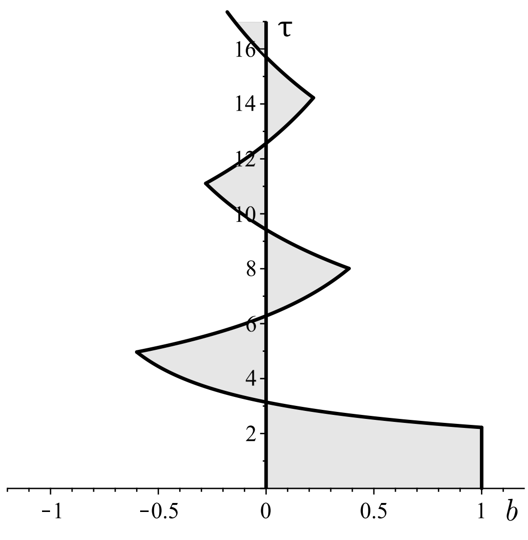

Figure 3.16: The stability region in (b,τ)-plane for $\alpha = 2$ and $a = -1$.

of a system of nonlinear inequalities) were given as a part of an example in [12].
(ii) Figure 3.15 depicts the stability region in the (b, τ)-plane showing clearly the delay-independent stability region, the stability switching property as well as the role of $\tau_j^{\pm}$ $(j \in \mathbb{Z}_0^+)$. Compare it to the Figures 3.8-3.10.

Let us consider $\alpha \to 2^-$ and compare it to the known results for (1.6). The asymptotes $p^+_{\alpha,m}$, $p^-_{\alpha,m}$ tend to the lines $b = -a - (m\pi)^2/\tau^2$ and $b = a + (m\pi)^2/\tau^2$, which are the lines forming the stability boundary of (1.6) (see Theorem 1.9). Taking $\alpha \to 2^-$ in Theorem 3.21, we obtain the limit for the stability region in the form: $0 < |b| < -a$ and

$$\frac{\ell\pi}{\sqrt{-a-|b|}} < \tau < \frac{(\ell+1)\pi}{\sqrt{-a+|b|}}$$

where ℓ is a nonnegative integer that is even for $b > 0$ and odd for $b < 0$ (see Figure 3.16). This form of conditions seems to be more effective compared to that of Theorem 1.9, especially with respect to explicit evaluations of stability switches for a varying delay parameter.

Chapter 4

Conclusions

In this thesis, we presented an in-depth exploration of the stability, asymptotic behaviour, and oscillatory properties of several linear fractional differential problems with a time delay. The focus was placed on optimal or near-optimal nature of achieved results and on their explicit form in terms of system parameters whenever possible. The introduced findings extend our understanding of nuances of fractional and classical dynamics and in many cases they pioneered the qualitative theory of fractional delay differential problems as outlined below.

We derived optimal stability conditions for the one-term FDDS of an arbitrary order (2.1), including a comprehensive analysis at the stability boundary (see Theorems 2.9 and 2.10). Rather unconventionally, we also dealt with detailed asymptotic description of unbounded solutions based on the location of system eigenvalues (see Theorems 2.13 and 2.14). While stability properties display a smooth transition across derivative orders, the asymptotic behaviour shows a striking contrast, as can be expected based on theory of undelayed fractional differential equations. In asymptotically stable cases, fractional derivatives lead to algebraic decay rates, as opposed to the exponential decay seen in classical systems. As a consequence, solutions to FDDS tend toward dominantly non-oscillatory behaviour which is a clear difference from their integer-order counterparts (see Theorem 2.12 and 2.16).

For the two-term FDDE (3.1) of orders less than two, we described the stability regions and grasped their evolution as the derivative order increases, passing through classical integer cases (see Theorems 3.4, 3.19). Moreover, we provided several insights into the emergence mechanism of stability switching phenomenon. Our particular focus was on providing stability criteria in practical form, i.e. in terms of entry coefficients, often in non-improvable versions (see Theorems 3.6, 3.14 and 3.21).

These theoretical results naturally transfer to practical applications, particularly to control theory. They outline effective design strategies for stabilization or destabilization of fractional systems via delayed feedback loops, as well as more nuanced prediction of large-time behaviour or such system (see [11,12]). In the future, we can expect emergence of other applications, e.g. in theory of complex systems where nonlocal and memory-based nature of fractional derivatives in combination with time lagging seems to be a promising direction.

Regarding future research, there are several promising directions extending stability and oscillatory analyses to more general cases. In particular, there are significant opportunities in areas where the author already has substantial experience in the undelayed context, such as discrete settings (see [7, 14, 34]), time-scale calculus (see [18, 33, 36]), nonlinear dynamics and variable coefficients (see [8, 35]), or problems including multiple fractional operators (see [9]).

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Appendices

Appendix A

Paper on lower-order one-term FDDS [6] (CNSNS, 2016)

Until 2016, I had already published nearly a dozen papers on fractional differential and difference equations. However, it was the paper [6] (co-authors: J. Čermák, J. Horníček; my author's share 45 %) that expanded my scope to include equations involving time delay. To this day, it remains my most cited work across all databases.

We focused on basics which were not sufficiently covered at the time. The stability and asymptotic properties of autonomous linear FDDS of order less than one. The main result was the formulation of necessary and sufficient conditions for stability via the location of system matrix eigenvalues in complex plane. We also derived algebraic decay rate of solutions tending to zero.

This paper laid the groundwork for techniques that we later employed for more advanced and technically challenging problems. Notably, we have re-established the fundamental solution for FDDS and introduced a generalized delay exponential function of Mittag-Leffler type. Most importantly, we adopted a technique utilizing the inverse Laplace transform and root analysis of the characteristic equation to derive asymptotic behaviour of solutions. This approach proved crucial for our subsequent research, as the presence of fractional derivatives disallows the direct use of the link between the real part of characteristic roots and argument of exponentials (since they do not belong among the solutions of fractional problems).

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Stability regions for fractional differential systems with a time delay

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Due to licensing restrictions, the full text of [6] is not included in the publicly accessible part of this habilitation thesis.

Appendix B

Paper on higher-order one-term FDDS [10] (EJDE, 2019)

We do not find direct technical generalizations of previous papers particularly interesting, which likely led us to postpone the work on higher-order one-term FDDS as it seemed like a simple follow-up on [6]. However, our hesitation proved unnecessary. While the stability results were indeed expectedly straightforward generalizations of our previous findings, [10] (co-author: J. Čermák; my author's share 50 %) shifted our focus towards the oscillatory properties - a challenge introduced by higher-order systems.

In this paper, we conducted a deeper analysis of the locations of characteristic roots depending on location of eigenvalues. Unlike common practice, we were not only interested in the case of negative real parts. We detailed the occurrence and conditions of roots with positive real parts, including their number. That started our interest in the properties of unbounded solutions, which we revisited also in subsequent papers.

Ultimately, [10] addresses FDDS of all positive non-integer orders, revealing predominantly non-oscillatory behaviour. To better discuss the mechanism by which initial conditions influence the oscillatory and stability properties of the given solution, we introduced the terms *major* and *n-minor* solutions. That allowed us to explore the effects of initial conditions more comprehensively.

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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. This article discusses the oscillatory and asymptotic properties of a test delay differential system involving a non-integer derivative order. We formulate corresponding criteria via explicit necessary and sufficient conditions that enable direct comparisons with the results known for classical integer-order delay differential equations. In particular, we shall observe that oscillatory behaviour of solutions of delay system with non-integer derivatives embodies quite different features compared to the classical results known from the integer-order case.

1. Introduction and preliminaries

Basic qualitative properties of the delay differential equation

$$y'(t) = Ay(t-\tau), \quad t \in (0,\infty), \tag{1.1}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag, are well described in previous numerous investigations. While stability and asymptotic properties of (1.1) were reported in [8], answers to various oscillation problems regarding (1.1) were surveyed in [7].

A crucial role in these investigations was played by the associated characteristic equation

$$\det(sI - A\exp\{-s\tau\}) = 0, \tag{1.2}$$

where I is the identity matrix. More precisely, appropriate properties of (1.1) were first described via location of all roots of (1.2) in a specific area of the complex plane. Then, efficient criteria guaranteeing such root locations were formulated in terms of conditions imposed directly on the eigenvalues of A.

We recall some of relevant statements (reformulated in the above mentioned sense) along with their consequences to the scalar case when (1.1) becomes

$$y'(t) = ay(t-\tau), \quad t \in (0,\infty) \tag{1.3}$$

where a is a real number. Since we are primarily interested in discussions of oscillatory properties of appropriate fractional extensions of (1.1), we first state (see [7]) oscillation conditions for (1.1) (as it is customary, we say that a solution of (1.1) is oscillatory if every its component has arbitrarily large zeros; otherwise the solution is called non-oscillatory).

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Theorem 1.1. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *All solutions of* (1.1) *oscillate;*
(b) *The characteristic equation* (1.2) *has no real roots;*
(c) *A has no real eigenvalues in $[-1/(\tau \mathrm{e}), \infty)$.*

Corollary 1.2. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. All solutions of* (1.3) *oscillate if and only if*

$$a < -\frac{1}{\tau \mathrm{e}}\,.$$

As we shall see later, oscillatory properties of the corresponding fractional delay system are closely related to convergence of all its solutions to the zero solution. In the first-order case (1.1), this property was characterized in [8] via

Theorem 1.3. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *Any solution y of* (1.1) *tends to zero as $t \to \infty$;*
(b) *The characteristic equation* (1.2) *has all roots with negative real parts;*
(c) *All eigenvalues λ_i $(i = 1, \dots, d)$ of A satisfy*

$$\tau|\lambda_i| < |\arg(\lambda_i)| - \pi/2\,.$$

Moreover, the convergence of y to zero is of exponential type.

Remark 1.4. The condition (c) can be equivalently expressed via the requirement that all eigenvalues λ_i $(i = 1, \dots, d)$ of A have to be located inside the region bounded by the curve

$$\Re(\lambda) = \omega\cos(\omega\tau), \quad \Im(\lambda) = -\omega\sin(\omega\tau), \quad -\frac{\pi}{2\tau} \le \omega \le \frac{\pi}{2\tau}$$

in the complex plane.

Corollary 1.5. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. Any solution y of* (1.3) *tends to zero as $t \to \infty$ if and only if*

$$-\frac{\pi}{2\tau} < a < 0\,.$$

Extensions of previous results to the n-th order equation (n is a positive integer)

$$y^{(n)}(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.4}$$

yield different conclusions. In this case, the characteristic equation becomes

$$\det(s^n I - A\exp\{-s\tau\}) = 0\,. \tag{1.5}$$

If $n \ge 2$, then there is no analogue to Theorem 1.3. More precisely, the convergence of all solutions of (1.4) to zero is not possible whenever $n \ge 2$ (see, e.g. [6]). Regarding oscillatory properties of (1.4), equivalency of conditions (a) and (b) (with (1.2) replaced by (1.5)) of Theorem 1.1 remains preserved, but their conversion into an explicit form depends on parity of n (see [7]).

The main goal of this article is to discuss these oscillatory and related asymptotic properties of (1.1) with respect to their possible extension to the fractional delay differential equation

$$\mathrm{D}_0^\alpha y(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.6}$$

where $\alpha > 0$ is a real scalar and the symbol D_0^α is the Caputo derivative of order α introduced in the following way: First let y be a real scalar function defined on $(0,\infty)$. For a positive real γ, the fractional integral of y is given by

$$\mathrm{D}_0^{-\gamma} y(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} y(\xi)\mathrm{d}\xi, \quad t \in (0,\infty)$$

and, for a positive real α, the Caputo fractional derivative of y is given by

$$\mathrm{D}_0^{\alpha} y(t) = \mathrm{D}_0^{-(\lceil\alpha\rceil-\alpha)} \Big(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} y(t)\Big), \quad t \in (0,\infty)$$

where $\lceil\cdot\rceil$ means the upper integer part. As it is customary, we put $\mathrm{D}_0^0 y(t) = y(t)$ (for more on fractional calculus, see, e.g. [10, 15]). If y is a real vector function, the corresponding fractional operators are considered component-wise (similarly, if y is a complex-valued function, then these fractional operators are introduced for its real and imaginary part separately). We add that the initial conditions associated to (1.6) are

$$y(t) = \phi(t), \quad t \in [-\tau, 0]\,, \tag{1.7}$$

$$\lim_{t\to 0^+} y^{(j)}(t) = \phi_j, \quad j = 0,\dots,\lceil\alpha\rceil - 1 \tag{1.8}$$

where all components of ϕ are absolutely Riemann integrable on $[-\tau, 0]$ and ϕ_j are real scalars. In the frame of our oscillatory and asymptotic discussions on (1.6), we are going not only to extend previous results to (1.6) but also discuss a dependence of relevant conditions on changing derivative order α (with a special attention to the case when α is crossing integer values).

The structure of this paper is following: Section 2 recalls some related special functions as well as the characteristic equation associated with (1.6). Some asymptotic expansions of the studied special functions are described as well. In Section 3, we discuss in detail distribution of roots of the characteristic equation in specific areas of the complex plane. Using these auxiliary statements, Sections 4 and 5 formulate a series of results describing oscillation and asymptotic properties of (1.6) in the vector and scalar case. More precisely, Section 4 presents analogues of Theorems 1.1 and 1.3, and Section 5 contains some additional oscillation results in the scalar case. Discussions on non-consistency of the obtained results with the above recalled classical properties of (1.1) and (1.3) are subject of Section 6 concluding the paper.

2. Special functions and their properties

In this section, we recall and extend some notions and formulae introduced in [3] in the frame of stability analysis of (1.6) with $0 < \alpha < 1$. As we shall see later, these tools turn out to be very useful also in oscillatory investigations of (1.6) with arbitrary real $\alpha > 0$. Since the proofs of auxiliary statements stated below are (essentially) analogous to the proofs of appropriate assertions from [3], we omit them.

In the sequel, the symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and inverse Laplace transform of appropriate functions, respectively.

Definition 2.1. Let $A \in \mathbb{R}^{d\times d}$, let I be the identity $d\times d$ matrix and let $\alpha, \tau \in \mathbb{R}^+$. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d\times d}$ given by

$$R(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}\right)(t) \tag{2.1}$$

is called the fundamental matrix solution of (1.6).

Theorem 2.2. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha, \tau \in \mathbb{R}^+$ and let R be the fundamental matrix solution of* (1.6)*. Then the solution y of* (1.6)–(1.8) *is given by*

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1} R(t)\phi_j + \int_{-\tau}^{0} R(t-\tau-u)A\phi(u)\mathrm{d}u.$$

Remark 2.3. Theorem 2.2 along with Definition 2.1 imply that the poles of the Laplace image of solution of (1.6) coincide with roots of

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0, \quad \text{equivalently} \quad \prod_{i=1}^{n}(s^\alpha - \lambda_i \exp\{-s\tau\})^{n_i} = 0\,, \tag{2.2}$$

where λ_i ($i = 1,\dots,n$) are distinct eigenvalues of A and n_i are their algebraic multiplicities. This confirms the well-known fact that (2.2) is the characteristic equation associated to (1.6) (see, e.g. [5, 9, 11]).

The following notion of a generalized delay exponential function plays an important role in description of asymptotic expansions of the fundamental matrix solution of (1.6).

Definition 2.4. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. The generalized delay exponential function (of Mittag-Leffler type) is introduced via

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty} \binom{m+j}{j} \frac{\lambda^j (t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)} h(t-(m+j)\tau)$$

where h is the Heaviside step function.

The relationship between the fundamental matrix solution R and the generalized delay exponential functions $G_{\eta,\beta}^{\lambda,\tau,m}$ can be specified via the following lemma.

Lemma 2.5. *The fundamental matrix solution* (2.1) *can be expressed as $R(t) = T^{-1}\mathcal{G}(t)T$, where T is a regular matrix and $\mathcal{G}$ is a block diagonal matrix with upper-triangular blocks B_j given by*

$$B_j(t) = \begin{pmatrix} G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,2}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-1}(t) \\ 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-2}(t) \\ 0 & 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) \end{pmatrix},$$

where $j = 1,\dots,J$ ($J \in \mathbb{Z}^+$) and r_j is the size of the corresponding Jordan block of A.

As a next key auxiliary result, we describe asymptotic behaviour of $G_{\eta,\beta}^{\lambda,\tau,m}$ functions.

Lemma 2.6. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. Further, let s_i ($i = 1, 2, \dots$) be the roots of*

$$s^\alpha - \lambda\exp\{-s\tau\} = 0 \tag{2.3}$$

with ordering $\Re(s_i) \ge \Re(s_{i+1})$ ($i = 1, 2, \dots$; in particular, s_1 is the rightmost root).

(i) *If* $\lambda = 0$, *then*

$$G^{0,\tau,m}_{\alpha,\beta}(t) = \frac{(t-m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha+\beta)} h(t-m\tau).$$

(ii) *If* $\lambda \neq 0$, *then*

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{m\cdot k_i} a_{ij}(t-m\tau)^j \exp\{s_i(t-m\tau)\} + S^{\lambda,\tau,m}_{\alpha,\beta}(t)\,,$$

where k_i *is a multiplicity of* s_i, a_{ij} *are suitable nonzero complex constants* ($j = 0,\dots,mk_i$, $i = 1,2,\dots$) *and the term* $S^{\lambda,\tau,m}_{\alpha,\beta}$ *has the algebraic asymptotic behaviour expressed via*

$$\begin{aligned} S^{\lambda,\tau,m}_{\alpha,\beta}(t) &= \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} \\ &\quad + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \quad \text{as } t\to\infty. \end{aligned}$$

3. Distribution of characteristic roots

The aim of this section is to analyse (2.2) with respect to existence of its real roots as well as number of its roots with positive real parts. Doing this, it is enough to consider its partial form (2.3).

First, we characterize the set of all roots of (2.3) in terms of their magnitudes and arguments (we assume here $\lambda \neq 0$, i.e. $s \neq 0$). Using the goniometric forms of s and λ we obtain that (2.3) is equivalent to

$$\begin{aligned} &|s|^\alpha \cos[\alpha \arg(s)] - |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}\cos[\arg(\lambda) - |s|\tau\sin(\arg(s))] \\ &= 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &|s|^\alpha \sin[\alpha \arg(s)] - |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}\sin[\arg(\lambda) - |s|\tau\sin(\arg(s))] \\ &= 0. \end{aligned} \tag{3.2}$$

To solve (2.3), we consider (3.1)–(3.2) as a system with unknowns $|s|$ and $\arg(s)$. If $\alpha\arg(s) = \ell_1\pi$ for some integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] = \ell_2\pi$ for some integer ℓ_2 and (3.1) yields

$$|s|^\alpha(-1)^{\ell_1} - |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}(-1)^{\ell_2} = 0,$$

i.e.

$$|s|^\alpha = (-1)^\ell|\lambda|\exp\{-|s|\tau\cos[\arg(s)]\} = 0 \quad \text{for some integer } \ell. \tag{3.3}$$

Thus (3.1)–(3.2) can be reduced to

$$\alpha\arg(s) - \arg(\lambda) - |s|\tau\sin[\arg(s)] = 2k\pi \quad \text{for some integer } k, \tag{3.4}$$

$$|s|^\alpha = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}. \tag{3.5}$$

If $\alpha\arg(s) \neq \ell_1\pi$ for any integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] \neq \ell_2\pi$ for any integer ℓ_2 and division (3.1) over (3.2) yields

$$\alpha\arg(s) = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\} + \ell\pi \quad \text{for some integer } \ell.$$

This, after substitution into (3.1), yields (3.3). Now, the same argumentation as above shows equivalency of (2.3) and (3.4)–(3.5).

Using the previous process, we can derive the following characterization of possible real roots of (2.3).

Proposition 3.1. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$.*

(i) *The characteristic equation* (2.3) *has a positive real root if and only if λ is a positive real. This root is simple, unique and it is the rightmost root of* (2.3)*.*

(ii) *The characteristic equation* (2.3) *has a negative real root if and only if*
$$0 < |\lambda| \le (\frac{\alpha}{\tau \mathrm{e}})^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}\,.$$
More precisely, if
$$0 < |\lambda| = (\frac{\alpha}{\tau \mathrm{e}})^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}\,,$$
then $s_{1,2} = -\alpha/\tau$ is double and the rightmost root of (2.3) *(remaining roots of* (2.3) *are not real). If*
$$0 < |\lambda| < \big(\frac{\alpha}{\tau \mathrm{e}}\big)^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \text{ for some } k \in \mathbb{Z}\,,$$
then (2.3) *has a couple of simple real negative roots, the greater of them being rightmost (remaining roots of* (2.3) *are not real).*

(iii) *The characteristic equation* (2.3) *has the zero root if and only if $\lambda = 0$.*

Furthermore, using (3.4)–(3.5) we can specify the distribution of characteristic roots of (2.3) with respect to the imaginary axis. Before doing this, we introduce the following areas in the complex plane.

For real parameters $0 < \alpha < 2$ and $\tau > 0$, we define the set $Q_0(\alpha, \tau)$ of all complex λ such that
$$|\arg(\lambda)| > \frac{\alpha\pi}{2} \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2}}{\tau}\Big)^{\alpha}.$$
Further, for any positive integer m and real parameters $0 < \alpha < 4m+2$ and $\tau > 0$, we define the sets $Q_m(\alpha, \tau)$ of all complex λ such that either
$$\frac{\alpha\pi}{2} - 2m\pi < |\arg(\lambda)| \le \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^{\alpha},$$
or $|\arg(\lambda)| > \frac{\alpha\pi}{2} - 2m\pi$ and
$$\Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + (2m-2)\pi}{\tau}\Big)^{\alpha} < |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^{\alpha}.$$
We add that the sets $Q_m(\alpha, \tau)$ $(m = 0, 1, \dots)$ are defined to be empty whenever $\alpha \ge 4m + 2$.

Now, we can describe the location of the roots of (2.3) with respect to the imaginary axis in terms of the sets $Q_m(\alpha, \tau)$ (we utilize here the standard notation $\partial[Q_m(\alpha, \tau)]$ for their boundaries).

Proposition 3.2. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$. Then there exist just m $(m = 0, 1, \dots)$ characteristic roots of* (2.3) *with a positive real part (while remaining roots have negative real parts) if and only if $\lambda \in Q_m(\alpha, \tau)$. Moreover,* (2.3) *has a root with the zero real part if $\lambda \in \partial[Q_m(\alpha, \tau)]$ for some $m = 0, 1, \dots$.*

The appropriate regions $Q_m(\alpha, \tau)$ are depicted in Figures 1 and 2.

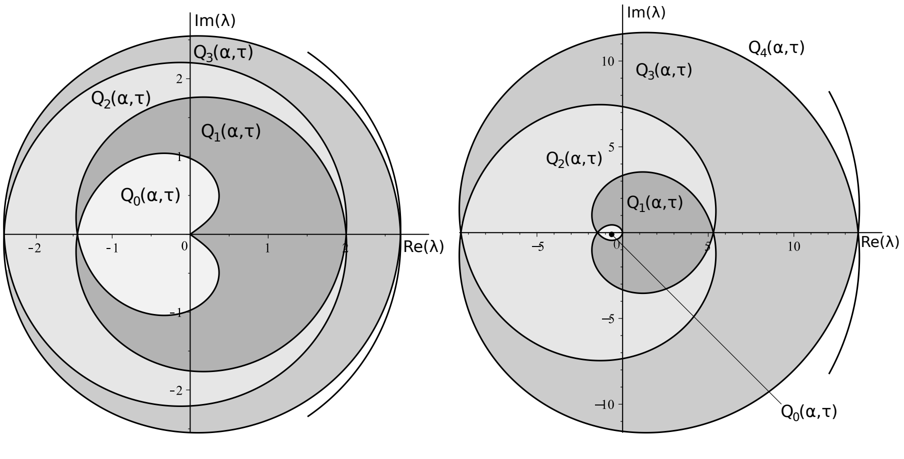

FIGURE 1. $\alpha = 0.4$ and $\tau = 1$ (left). $\alpha = 1.1$ and $\tau = 1$ (right)

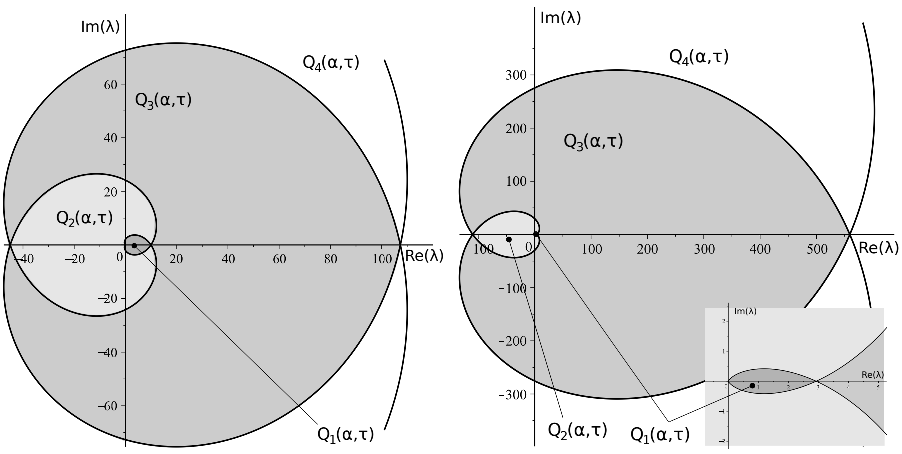

FIGURE 2. $\alpha = 2.1$ and $\tau = 1$ (left). $\alpha = 3.1$ and $\tau = 1$ (right)

Proof. We start with the proof of Proposition 3.1 and consider the characterization of roots s of (2.3) via (3.4)–(3.5). Obviously, (2.3) has a positive real root if $\arg(\lambda) = 0$ (i.e. λ is a positive real). In this case, the characteristic function

$$F(s) = s^\alpha - \lambda \exp\{-s\tau\}$$

is strictly increasing for all $s \geq 0$ with $F(0) = -\lambda < 0$ and $F(\infty) = \infty$, hence there is a unique positive real root s_1 of (2.3). To show its dominance, we consider remaining roots s_i of (2.3) with a positive real parameter λ. Then (3.5) yields

$$(s_1)^\alpha = \lambda \exp\{-s_1\tau\}, \quad |s_i|^\alpha = \lambda \exp\{-|s_i|\tau \cos[\arg(s_i)]\}.$$

From here, we obtain

$$\Big(\frac{s_1}{|s_i|}\Big)^\alpha = \exp\{(-s_1 + |s_i| \cos[\arg(s_i)])\tau\}\,. \tag{3.6}$$

Assume that s_1 is not the rightmost root of (2.3), i.e. $|s_i|\cos[\arg(s_i)] \geq s_1$ for some root s_i of (2.3). Then

$$\frac{s_1}{|s_i|} < 1 \quad \text{and} \quad \exp\{(-s_1 + |s_i|\cos[\arg(s_i)])\tau\} \geq 1$$

which contradicts (3.6). This proves Proposition 3.1(i).

Similarly, (3.4)–(3.5) imply that (2.3) has a negative real root s if and only if

$$\arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}$$

and

$$|s|^\alpha = |\lambda| \exp\{|s|\tau\}.$$

Put $r = |s|$ and $G(r) = r^\alpha - |\lambda|\exp\{r\tau\}$, $r \geq 0$. Then $G(0) = -|\lambda| < 0$, $G(\infty) = -\infty$ and G is increasing in $(0, r^*)$ and decreasing in (r^*, ∞) for a suitable $r^* > 0$. Thus G has (one or two) positive roots if and only if $G(r^*) \geq 0$. In particular, G has a unique positive root r^* if and only if $G(r^*) = G'(r^*) = 0$, i.e.

$$(r^*)^\alpha - |\lambda|\exp\{r^*\tau\} = \alpha(r^*)^{\alpha-1} - |\lambda|\tau\exp\{r^*\tau\} = 0\,.$$

From here, we obtain

$$r^* = \frac{\alpha}{\tau} \quad \text{and} \quad |\lambda| = \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha.$$

Obviously, if

$$|\lambda| < \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha,$$

then G has two real positive roots $r_1 < r_2$. We show that $s_1 = -r_1$ is the rightmost root of (2.3), i.e $s_1 > |s_i|\cos[\arg(s_i)]$ for all remaining roots s_i $(i = 2, 3, \dots)$ of (2.3). Indeed, by (3.5),

$$|s_1|^\alpha = |\lambda|\exp\{|s_1|\tau\} \quad \text{and} \quad |s_i|^\alpha = |\lambda|\exp\{-|s_i|\tau\cos[\arg(s_i)]\}\,.$$

Then $|s_1| < |s_i|$, i.e. $|s_1| + |s_i|\cos[\arg(s_i)] < 0$. Analogously we can show the dominance of a double real root $s_{1,2}$ (if exists). This proves Proposition 3.1 (ii). The assertion of Proposition 3.1(iii) is trivial.

Now, we show the validity of Proposition 3.2. Since the case of real characteristic roots of (2.3) has been discussed previously, we first search the roots s with $0 < \arg(s) \leq \pi/2$. Then (3.4)–(3.5) can be reduced to

$$|s| = \frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}, \tag{3.7}$$

$$\begin{aligned}\Big(\frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}\Big)^\alpha - |\lambda|\exp\big\{(-\arg(\lambda)\\ + \alpha\arg(s) - 2k\pi)\ \mathrm{cotan}[\arg(s)]\big\} = 0\,.\end{aligned} \tag{3.8}$$

We denote the left-hand side of (3.8) by $H_k = H_k(\arg(s))$. Then

$$H_k(0^+) = \infty, \quad H_k(\pi/2) = \Big(\frac{\arg(\lambda) - \alpha\pi/2 + 2k\pi}{\tau}\Big)^\alpha - |\lambda|$$

and $H_k(\arg(s))$ decreases as $\arg(s)$ increases from 0 to $\pi/2$. This implies that (3.7)–(3.8) has just m couples of solutions with $|s| > 0$ and $0 < \arg(s) \leq \pi/2$ if and only if either

$$\frac{\alpha\pi}{2} - 2m\pi < \arg(\lambda) \leq \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad H_m(\pi/2) > 0\,,$$

or

$$\arg(\lambda) > \frac{\alpha\pi}{2} - 2m\pi \quad \text{and} \quad H_m(\pi/2) > 0 > H_{m-1}(\pi/2)\,.$$

If $-\pi/2 \le \arg(s) < 0$, then we obtain the same conclusion with $\arg(\lambda)$ replaced by $-\arg(\lambda)$. This implies the main part of the assertion. The supplement on existence of purely imaginary roots of (2.3) follows from continuous dependence of roots s on parameter λ. Alternatively, it can be obtained via the standard D-decomposition method. □

4. Main results

In this section, we derive and formulate fractional-order analogues to Theorems 1.1 and 1.3.

Theorem 4.1. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

- (a) *All non-trivial solutions of* (1.6) *are non-oscillatory;*
- (b) *The characteristic equation* (2.2) *admits only real roots or roots with a negative real part;*
- (c) *A has all eigenvalues lying in $Q_0(\alpha,\tau) \cup (Q_1(\alpha,\tau)\cap\mathbb{R}) \cup \{0\}$.*

Proof. Theorem 2.2 and Lemma 2.5 imply that every solution of (1.6)–(1.8) can be expressed as

$$y(t) = T^{-1}\sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)T\phi_j + T^{-1}\int_{-\tau}^{0}\mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u\,, \tag{4.1}$$

where $\mathcal{G}$ is a matrix function introduced in Lemma 2.5, J is a Jordan form of the system matrix A and T is the corresponding regular projection matrix, i.e. $A = TJT^{-1}$. Employing (4.1) and Lemma 2.5, we can see that every component of y is a linear combination of terms derived from elements of $\mathcal{G}$. We distinguish two cases with respect to (non)zeroness of eigenvalues λ_i of A.

First, let $\lambda_i \neq 0$ for all $i = 1,\dots,n$ (n being the number of distinct eigenvalues of A). Then the elements of matrices $\mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)$ ($j = 0,\dots,\lceil\alpha\rceil-1$) can be asymptotically expanded via the relation

$$\begin{aligned}
&\mathrm{D}_0^{\alpha-j-1}G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t) = G_{\alpha,j+1}^{\lambda_i,\tau,m}(t)\\
&= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^\ell \exp\{s_w t\} b_{w,\ell}\big(1-\frac{m\tau}{t}\big)^\ell \exp\{-s_w m\tau\}\\
&\quad + t^{j-\alpha}\frac{(-1)^{m+1}(1+\tau/t)^{j-\alpha}}{\lambda_i^{m+1}\Gamma(j-\alpha+1)} + \mathcal{O}(t^{j-2\alpha}) \quad \text{as } t\to\infty\,,
\end{aligned} \tag{4.2}$$

where s_w ($w = 1,2,\dots,N$) are roots of (2.3) with the largest real parts ordered as $\Re(s_w) \ge \Re(s_{w+1})$, N is any positive integer satisfying $\Re(s_N) < 0$, k_w is multiplicity of s_w and $b_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)). Similarly, the elements of the matrix $\int_{-\tau}^{0} \mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u$ have the expansions

$$\begin{aligned}
&\int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u \\
&= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^\ell \exp\{s_w t\} c_{w,\ell}\lambda_i \int_{-\tau}^{0}\Big(1-\frac{(m+1)\tau}{t}-\frac{u}{t}\Big)^\ell \mathrm{e}^{-s_w((m+1)\tau+u)}\hat{\phi}^p(u)\mathrm{d}u \\
&\quad + t^{-\alpha-1}\int_{-\tau}^{0}\frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)}\hat{\phi}^p(u)\mathrm{d}u + \mathcal{O}(t^{-2\alpha-1})
\end{aligned} \tag{4.3}$$

as $t \to \infty$, where $\hat{\phi}^p(u)$ is pth row of the vector $JT\phi(u)$ and $c_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)).

If $\lambda_i = 0$ for some $i = 1,\dots,n$, then the appropriate analogues of (4.2)–(4.3) involve only algebraic terms (see Lemma 2.6(i)). Now, we can prove the presented equivalencies:

(a)⇔(b): The property (a) holds if and only if, for any choice of ϕ, the dominating terms involved in (4.2) and (4.3) are non-oscillatory. We can see that all the algebraic terms from (4.2) and (4.3) are non-oscillatory and eventually dominating with respect to all exponential terms with negative real parts of their arguments. Contrary, an exponential term is eventually dominating provided its argument has a non-negative real part. Clearly, if such a case does occur, the solution y of (1.6) is non-oscillatory only if the imaginary parts of the corresponding arguments are zero. By (4.2) and (4.3), the discussed arguments of the exponential terms are expressed via roots of (2.2), which yields equivalency of (a) and (b).

(b)⇔(c): This equivalency follows immediately from Propositions 3.1 and 3.2. □

In the scalar case, when (1.6) becomes

$$\mathrm{D}_0^\alpha y(t) = ay(t-\tau), \quad t \in (0,\infty)\,, \tag{4.4}$$

a being a real scalar, we obtain the following explicit characterization of non-existence of a non-trivial oscillatory solution.

Corollary 4.2. *Let $a \in \mathbb{R}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. All non-trivial solutions y of* (4.4) *are non-oscillatory if and only if*

$$0 < \alpha < 2 \quad \textit{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^\alpha < a < \Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^\alpha,$$

or

$$2 < \alpha < 4 \quad \textit{and} \quad 0 < a < \Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^\alpha.$$

Remark 4.3. In the first-order case, the value $a = -1/(\tau\mathrm{e})$ is of a particular importance: crossing this value, the (negative) real roots of the associated characteristic equation disappear and all solutions of (1.3) become oscillatory for $a < -1/(\tau\mathrm{e})$. In the fractional-order case, the (negative) real roots disappear for $a < -(\alpha/(\tau\mathrm{e}))^\alpha$. However, such roots have no impact on oscillatory behaviour of the solutions of (4.4) because the exponential terms with negative arguments involved in the formulae (4.1)–(4.3) are eventually suppressed by algebraic terms.

By Theorem 4.1, if all roots of (2.2) have negative real parts, then all non-trivial solutions of (1.6) are non-oscillatory. Therefore, we give an explicit characterization of this assumption and thus provide a fractional-order analogue to Theorem 1.3.

Theorem 4.4. *Let $A \in \mathbb{R}^{d\times d}$ and $\alpha, \tau \in \mathbb{R}^+$. Then the following statements are equivalent*

(a) *Any solution y of* (1.6) *tends to zero as $t \to \infty$;*
(b) *The characteristic equation* (2.2) *has all roots with negative real parts;*
(c) *All eigenvalues λ_i $(i = 1, \dots, d)$ of A are nonzero and satisfy*

$$\tau|\lambda_i|^{1/\alpha} < |\arg(\lambda_i)| - \alpha\pi/2\,.$$

Moreover, if $\alpha \notin \mathbb{Z}^+$, then the convergence to zero is of algebraic type; more precisely, for any solution y of (1.6) *there exists a suitable integer $j \in \{0, \dots, \lceil\alpha\rceil\}$ such that $|y(t)| \sim t^{j-\alpha-1}$ as $t \to \infty$ (the symbol $\sim$ stands for asymptotic equivalency).*

Proof. (a)⇔(b): If $\lambda_i = 0$ for some $i = 1, \dots, d$, then the appropriate analogues of (4.2) and (4.3) yield that there is always a constant term involved in these expansions (this constant is nonzero if ϕ_0 is nonzero), hence the property (a) is not true. Obviously, the property (b) cannot occur as well provided $\lambda_i = 0$ for some $i = 1, \dots, d$. Thus, without loss of generality, we may assume $\lambda_i \neq 0$ for all $i = 1, \dots, d$.

The statement (a) is valid if and only if (4.2) and (4.3) do not contain any terms with a non-negative real part of the argument, which directly yields the equivalency (see also [11]).

(b)⇔(c): It is a direct consequence of Proposition 3.2.

Consequently, since all the exponential terms in (4.2) and (4.3) have a negative argument, they are suppressed by the algebraic terms. The presence of the term behaving like $t^{j-\alpha-1}$ for $j = 1, \dots, \lceil\alpha\rceil$ as $t \to \infty$ is determined by values ϕ_{j-1}. If $\phi_{j-1} = 0$ for all $j = 1, \dots, \lceil\alpha\rceil$, the integral term (4.3) becomes dominant. The integrability of ϕ enables us to write

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t^{-\alpha-1}} \Big| \int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u \Big| \\
&= \Big| \int_{-\tau}^{0} \lim_{t\to\infty} \frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)} \hat{\phi}^p(u)\mathrm{d}u \Big| \\
&= K \Big| \int_{-\tau}^{0} \hat{\phi}^p(u)\mathrm{d}u \Big|
\end{aligned}$$

for a suitable real K, therefore the integral term behaves like $t^{-\alpha-1}$ as $t \to \infty$. This completes the proof. □

For the case of scalar equation (4.4), we obtain the following result.

Corollary 4.5. *Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. All solutions y of* (4.4) *tend to zero if and only if*

$$\alpha < 2 \quad \text{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^{\alpha} < a < 0\,.$$

In particular, an interesting link between Theorems 4.1 and 4.4 is provided by the following assertion.

Corollary 4.6. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. If* (1.6) *has a nontrivial oscillatory solution, then it has also a solution which does not tend to zero as $t \to \infty$.*

Remark 4.7. In fact, formulae (4.1)–(4.3) reveal that any non-trivial solution of (1.6) tending to zero is non-oscillatory. Moreover, the solutions tending to zero pose an algebraic decay (there is no solution with an exponential decay).

5. Other oscillatory properties of (4.4)

In the classical integer-order case, oscillation argumentation often uses the fact that $\exp(s_w t)$ is a solution of (1.3) for any root s_w of the corresponding characteristic equation

$$s - a\exp\{-s\tau\} = 0\,. \tag{5.1}$$

In particular, if (5.1) admits a real root, then (1.3) has (via appropriate choice of ϕ) a non-oscillatory solution. In the fractional-order case, no such a direct connection for the influence study of characteristic roots of

$$s^\alpha - a\exp\{-s\tau\} = 0 \tag{5.2}$$

on the oscillatory behaviour of (4.4) is available. Nevertheless, as we can see from (4.1)–(4.3), the exponential functions generated by characteristic roots of (5.2) again play an important role in qualitative analysis of solutions of (4.4). Using this fact, we are able to describe some oscillatory properties of (4.4) with respect to asymptotic relationship between the studied solutions and the corresponding exponential functions. To specify this relationship, we introduce the following asymptotic classifications of solutions of (4.4).

Definition 5.1. Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. The solution y of (4.4) is called major solution, if it satisfies the asymptotic relationship

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_1}\exp\{s_1 t\}}\Big| > 0\,,$$

where s_1 is the rightmost root of (5.2) and k_1 its algebraic multiplicity.

Definition 5.2. Let $a \in \mathbb{R}$, $\alpha, \tau \in \mathbb{R}^+$, s_w $(w = 1, 2, \dots)$ be roots of (5.2) with ordering $\Re(s_w) \ge \Re(s_{w+1})$ and let k_w $(w = 1, 2, \dots)$ be the corresponding algebraic multiplicities. The solution y of (4.4) is called m-minor solution, if it satisfies the asymptotic relationships

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_m}\exp\{s_m t\}}\Big| = 0 \quad \text{and} \quad \limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_{m+1}}\exp\{s_{m+1} t\}}\Big| > 0\,.$$

Remark 5.3. The notions of the major and m-minor solutions are not just theoretical, but such solutions can be constructively obtained via appropriate choice of the initial function ϕ. For example, if s_1 is simple with a non-negative real part, then, by (4.1)–(4.3), the major solution occurs if ϕ meets the condition

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{1,j} + ac_{1,0}\int_{-\tau}^{0} \phi(u)\exp\{-s_1(\tau+u)\}\mathrm{d}u \neq 0$$

where $b_{1,j}$, $c_{1,0}$ have the same meaning as in (4.2)–(4.3). Clearly, such a condition is satisfied by infinitely many initial functions, e.g. by $\phi(u) = 1$, $\phi_j = 0$ $(j = 1, \dots, \lceil\alpha\rceil - 1)$ and $\phi_0 \neq -ac_{1,0}(1 - \exp\{-s_1\tau\})/(b_{1,0}s_1)$. Similarly, m-minor solution is characterized by the conditions

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{w,j} + ac_{w,0}\int_{-\tau}^{0} \phi(u)\exp\{-s_w(\tau+u)\}\mathrm{d}u = 0 \quad \text{for } w = 1, \dots, m\,,$$

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{m+1,j} + ac_{m+1,0} \int_{-\tau}^{0} \phi(u) \exp\{-s_{m+1}(\tau+u)\} \mathrm{d}u \neq 0$$

provided s_w $(w = 1, \dots, m+1)$ are simple roots and $b_{w,j}$, $c_{w,0}$, $b_{m+1,j}$, $c_{m+1,0}$ have the same meaning as in (4.2)–(4.3).

Using the notions of major and m-minor solutions, we can formulate in a more detail assertions revealing the relation between oscillatory properties of (4.4) and location of roots of (5.2) in the complex plane.

Lemma 5.4. *Let $a \in \mathbb{R} \setminus (Q_0(\alpha,\tau) \cup \{0\})$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let s_w $(w = 1, 2, \dots)$ be roots of (5.2) with ordering $\Re(s_w) \geq \Re(s_{w+1})$. Then the major solutions of (4.4) do not tend to zero and there exists $M > 0$ such that all m-minor solutions of (4.4) are non-oscillatory and tend to zero as $t \to \infty$ for all $m \geq M$. Furthermore, it holds:*

- (i) *If $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$, then all major solutions of (4.4) are oscillatory.*
- (ii) *If $\alpha < 4$ and $0 < a < ((4-\alpha)\pi/(2\pi))^\alpha$, then all non-trivial solutions of (4.4) are non-oscillatory.*
- (iii) *If $\alpha < 4$ and $a = ((4-\alpha)\pi/(2\pi))^\alpha$, then all major solutions of (4.4) are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and bounded.*
- (iv) *If $a > ((4-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 4$ or $a > 0$ for $\alpha > 4$, then all major solutions of (4.4) are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and unbounded.*

Proof. The first part of the assertion follows from the expansion of solution y of (4.4) based on (4.2)–(4.3). By Proposition 3.2, the rightmost root s_1 has a non-negative real part, therefore the major solutions involve, as a dominant term, an exponential function which does not tend to zero. Using a technique similar to that in Remark 5.3 we can always eliminate all terms in the asymptotic expansion of y corresponding to the characteristic roots with a non-negative real part, and, thus, construct non-oscillatory m-minor solutions algebraically tending to zero. Further utilization of this arguments enables us to obtain even more detailed results:

(i) The value $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$ guarantees that the rightmost root s_1 has a non-negative real part and non-zero imaginary part (see Propositions 3.1 and 3.2), therefore the major solutions are oscillatory.

(ii)–(iv) If $a > 0$, Proposition 3.1(i) implies that the rightmost root s_1 is a positive real, therefore the major solutions are non-oscillatory. Eliminating the rightmost root s_1 as in Remark 5.3, the terms corresponding to s_2 become dominant and, again using Proposition 3.2, we obtain the parts (ii)–(iv). □

Remark 5.5. For $a = 0$, (5.2) has the only root $s_1 = 0$ with multiplicity $\lceil\alpha\rceil$ and the qualitative behaviour is implied directly by Lemma 2.6(i). In particular, if $\alpha < 1$, then all non-trivial solutions of (4.4) are constant, i.e. they are bounded and non-oscillatory. If $\alpha > 1$, then all non-trivial solutions of (4.4) are non-oscillatory. Moreover, if $\phi_j = 0$ for all $j = 1, \dots, \lceil\alpha\rceil - 1$, then the solutions are bounded, otherwise being unbounded.

It is of a particular interest to emphasize that unlike the integer-order case, there is no combination of entry parameters such that all the solutions of (4.4) are oscillatory. In fact, (4.4) has always infinitely many non-oscillatory solutions.

6. Concluding remarks

We have discussed oscillatory and related asymptotic properties of solutions of the fractional delay differential system (1.6) as well as of the corresponding scalar equation (4.4). The obtained oscillation results qualitatively differ from those known from the classical oscillation theory of (integer-order) delay differential equations. We survey here the most important notes related to this phenomenon.

First, while the appropriate criteria from the classical theory (such as Theorem 1.1) formulate necessary and sufficient conditions for oscillation of all solutions, their fractional counterparts (Theorem 4.1) present conditions for non-oscillation of all non-trivial solutions. In particular, our analysis shows that (1.6) cannot admit only oscillatory solutions. Secondly, considering (1.6), one can observe a close resemblance between non-oscillation of all non-trivial solutions and convergence to zero of all solutions (this property defines asymptotic stability of the zero solution of (1.6)). The latter property is sufficient for non-oscillation of all non-trivial solutions of (1.6) and, moreover, it is not far from being also a necessary one. These features (along with some other precisions made in Section 5) demonstrate that (non)oscillatory properties of (1.6) qualitatively depend on the fact if the value α is integer or non-integer. In particular, Corollary 4.2 implies that the endpoints of corresponding non-oscillation intervals depend continuously on changing non-integer derivative order α; when α is crossing the integer-order value, a sudden change in oscillatory behaviour occurs (see Corollary 1.2). Note that despite of some introductory papers on oscillation of (1.6) and other related fractional delay differential equations (see, e.g. [1, 17]), these properties have not been reported yet.

On the other hand, one can observe that dependence of stability areas of (1.6) on changing derivative order is "continuous". As illustrated via Figures 1–4, this area is continuously becoming smaller, starting from the circle (corresponding to the non-differential case when $\alpha = 0$) to the empty set (when $\alpha = 2$). We add that the way to stability remains closed for all real $\alpha \geq 2$. From this viewpoint, considerations of (1.6) with non-integer derivative order enable a better understanding of classical stability results on (1.6) with integer α.

The method utilized in our oscillation analysis indicates that the main reason of a rather strange oscillatory behaviour of (1.6) with non-integer α is hidden in the algebraic rate of convergence of its solutions to zero (compared to the exponential rate known from the integer-order case). Since this type of convergence has been earlier described not only for other types of fractional delay equations (see [2, 9, 11, 12]), but also for fractional equations without delay (see [4, 13, 14, 16]), the above described oscillatory behaviour might be typical for a more general class of fractional differential equations.

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Appendix C

Paper on overview for one-term FDDS [37] (Math Appl, 2020)

Our work on [6,10] brought us significant insights into the stability and asymptotics of one-term FDDS, but not all findings aligned with the concepts of previous papers. Consequently, [37] (my author's share 100 %) aimed to consolidate the topic and to extend it.

In this paper, we provided a comprehensive overview of the stability and asymptotics theory for one-term FDDS, considering the two most common definitions of fractional derivatives: Caputo and Riemann-Liouville. Building on techniques adopted in our prior research, we derived optimal stability conditions based on the position of eigenvalues in the complex plane. We elaborated on the implications of using different definitions of fractional derivative, detailing distinctions on the stability boundary and in overall asymptotic behaviour.

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ON STABILITY OF DELAYED DIFFERENTIAL SYSTEMS OF ARBITRARY NON-INTEGER ORDER

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Abstract. This paper summarizes and extends known results on qualitative behavior of solutions of autonomous fractional differential systems with a time delay. It utilizes two most common definitions of fractional derivative, Riemann–Liouville and Caputo one, for which optimal stability conditions are formulated via position of eigenvalues in the complex plane. Our approach covers differential systems of any non-integer orders of the derivative. The differences in stability and asymptotic properties of solutions induced by the type of derivative are pointed out as well.

1. Introduction

In many areas of science and technology we often meet problems which are well described by differential systems with a time delay. Examples of such situations might be reaction time of technical and chemical systems or heredity in population dynamics. Qualitative theory for these equations is summarized in, e.g. [2,5]. The study of delayed systems involving viscoelasticity, anomalous diffusion or control theory naturally suggests to enrich our models with derivatives of non-integer order which proved to be very effective in these areas (see, e.g. [4,8]).

This is the main motivation for our study of two delayed systems which can be written as

$$\mathbf{D}_0^{\alpha} y(t) = Ay(t-\tau)\,, \quad t \in (0,\infty),\ \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}, \tag{1.1}$$

$$y(t) = \phi(t)\,, \quad t \in [-\tau, 0], \tag{1.2}$$

$$\mathbf{D}_0^{\alpha-k} y(t)\big|_{t=0} = y_{\alpha-k}\,, \quad k = 1, \dots, \lceil\alpha\rceil \tag{1.3}$$

and

$${}^{C}\mathrm{D}_0^{\alpha} y(t) = Ay(t-\tau)\,, \quad t \in (0,\infty),\ \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}, \tag{1.4}$$

$$y(t) = \phi(t)\,, \quad t \in [-\tau, 0], \tag{1.5}$$

$$y^{(\lceil\alpha\rceil-k)}(0) = y_{\lceil\alpha\rceil-k}\,, \quad k = 1, \dots, \lceil\alpha\rceil\,, \tag{1.6}$$

where $\mathbf{D}_0^{\alpha}$ and ${}^{C}\mathrm{D}_0^{\alpha}$ denote the so-called Riemann–Liouville and Caputo differential operators of order α, respectively. Further, $A \in \mathbb{R}^{d\times d}$ is a constant $d \times d$ matrix, $y_{.} \in \mathbb{R}^d$ are constant vectors and $\tau > 0$ is a constant delay. As usual for delayed

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equations, the initial condition is given by $\phi \in L^1[-\tau, 0]$ (componentwise) and the use of fractional derivatives allows us to prescribe also initial values for $t = 0$ separately. We intentionally leave out the integer-order values of α since they coincide with the known classical cases.

A serious qualitative analysis of such equations is being performed less than two decades. It spans across scalar and vector cases, various methods like D-decomposition or Laplace transform are used. For more details we refer to [1,3,6,9] which are the main sources for this paper.

The paper is organized as follows. In Section 2 we outline some basic preliminary results useful in our further considerations. Section 3 is devoted to the discussion of solution representations and their comparison. The main results are concentrated in Section 4 where we summarize known facts as well as derive some original ones. Section 5 concludes the paper with a few final remarks.

2. Preliminaries

Let f be a real function. We use the standard definition of fractional integral of order $\gamma > 0$

$$\mathrm{I}_0^\gamma f(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} f(\xi)\mathrm{d}\xi, \quad t \geq 0\,.$$

We employ both the wide used definitions of fractional derivative of order $\alpha > 0$ called the Riemann-Liouville and Caputo derivative introduced as

$$\begin{aligned} \mathbf{D}_0^\alpha f(t) &= \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} \left(\mathrm{I}_0^{\lceil\alpha\rceil-\alpha} f(t) \right), \quad t \geq 0\,, \\ {}^C\mathrm{D}_0^\alpha f(t) &= \mathrm{I}_0^{\lceil\alpha\rceil-\alpha} \left(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} f(t) \right), \quad t \geq 0\,, \end{aligned}$$

respectively. Additionally, we put ${}^C\mathrm{D}_0^0 f(t) = \mathbf{D}_0^0 f(t) = f(t)$ (for more information on fractional operators we refer, e.g. to [4, 8]).

The key tool, utilized throughout this paper, is the Laplace transform which is, for f, introduced as

$$\mathcal{L}(f(t))(s) = \int_0^\infty \exp\{-st\} f(t)\mathrm{d}t, \quad s \in \mathbb{C}$$

provided the integral converges. To perform the transform of (1.1) and (1.4), we need a clear view on Laplace transform of a function with shifted (delayed) argument which is given by

$$\begin{aligned} &\mathcal{L}(f(t-\tau)h(t-\tau))(s) = \exp\{-\tau s\}\mathcal{L}(f(t))(s), \quad \tau > 0, \\ &\mathcal{L}(f(t-\tau))(s) = \exp\{-\tau s\}\mathcal{L}(f(t))(s) + \exp\{-\tau s\} \int_{-\tau}^0 \exp\{-st\} f(t)\mathrm{d}t, \quad \tau > 0\,. \end{aligned} \tag{2.1}$$

Also, using the formulae for Laplace transform of convolution and power function

$$\mathcal{L}\left(\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi\right)(s) = \mathcal{L}(f(t))(s)\cdot\mathcal{L}(g(t))(s),$$
$$\mathcal{L}\left(\frac{t^\eta}{\Gamma(\eta+1)}\right)(s) = s^{-\eta-1}, \quad \eta > -1,$$

we can see the origin of Laplace transforms of fractional operators

$$\mathcal{L}(\mathrm{I}_0^\gamma f(t))(s) = s^{-\gamma}\mathcal{L}(f(t))(s), \quad \gamma > 0,$$
$$\mathcal{L}(\mathbf{D}_0^\alpha f(t))(s) = s^\alpha\mathcal{L}(f(t))(s) - \sum_{k=1}^{\lceil\alpha\rceil} s^{k-1}\mathbf{D}_0^{\alpha-k}f(t)\big|_{t=0}, \quad \alpha > 0, \tag{2.2}$$
$$\mathcal{L}({}^C\mathrm{D}_0^\alpha f(t))(s) = s^\alpha\mathcal{L}(f(t))(s) - \sum_{k=1}^{\lceil\alpha\rceil} s^{\alpha-k}f^{(k-1)}(0), \quad \alpha > 0. \tag{2.3}$$

The symbol h denotes the Heaviside step function defined as $h(\xi) = 1$ for $\xi \geq 0$ and $h(\xi) = 0$ for $\xi < 0$. When applied on a vector function, the Laplace transform is considered componentwise.

We note that the system matrix A of (1.1) and (1.4) can be rewritten with the use of a matrix Λ in a Jordan canonical form with the Jordan blocks on its diagonal as $A = T\Lambda T^{-1}$, where T is a regular real $d\times d$ matrix,

$$\Lambda = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_q \end{pmatrix}, \quad J_k = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & \lambda_i & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{pmatrix}, \quad k = 1,\dots,q,$$

and λ_i $(i = 1,\dots,n)$ are distinct eigenvalues of A. The number of Jordan blocks corresponding to λ_i is called geometric multiplicity of λ_i. The sum of the sizes of all Jordan blocks corresponding to λ_i is called algebraic multiplicity of λ_i.

Before we proceed to the next section, we recall the stability notions related to our linear fractional differential systems with a delay. The zero solution is said to be stable (asymptotically stable) if the solution of the system is bounded (tends to zero as $t\to\infty$) for any initial function $\phi \in L^1([-\tau,0])$.

3. Solution representations for (1.1) and (1.4)

As in the integer-order case (see, e.g. [2,5]), an essential role is played by analogue of the fundamental matrix solution also for (1.1) and (1.4) (see, e.g. [1]). In order to simplify the notation dealing with the orders α greater than one, we introduce its generalization in form of the following functions

$$R_{\alpha,\beta}^{A,\tau}(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}s^{\alpha-\beta}\right)(t), \quad \alpha\in\mathbb{R}^+\setminus\mathbb{Z},\ \beta\in\mathbb{R}^+$$

where $A\in\mathbb{R}^{d\times d}$ and I is the identity $d\times d$ matrix. Employing these R-functions, we arrive at the following solution representations.

Theorem 3.1. *The solution y_{RL} of (1.1)–(1.3) is given by*

$$y_{RL}(t) = \sum_{k=1}^{\lceil\alpha\rceil} R^{A,\tau}_{\alpha,\alpha-k+1}(t) y_{\alpha-k} + \int_{-\tau}^{0} R^{A,\tau}_{\alpha,\alpha}(t-\tau-u) A\phi(u)\mathrm{d}u.$$

Proof. Applying (2.1), (2.2) on (1.1)–(1.3), we get

$$\begin{aligned}
&\mathcal{L}(y(t))(s) \\
&= (s^\alpha I - A\exp\{-s\tau\})^{-1} \left[\sum_{k=1}^{\lceil\alpha\rceil} s^{k-1} y_{\alpha-k} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\} A\phi(t)\mathrm{d}t\right] \\
&= \sum_{k=1}^{\lceil\alpha\rceil} \mathcal{L}(R^{A,\tau}_{\alpha,\alpha-k+1}(t))(s) y_{\alpha-k} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\} \mathcal{L}(R^{A,\tau}_{\alpha,\alpha}(t))(s) A\phi(t)\mathrm{d}t
\end{aligned}$$

which yields the assertion. □

Theorem 3.2. *The solution y_C of (1.4)–(1.6) is given by*

$$y_C(t) = \sum_{k=1}^{\lceil\alpha\rceil} R^{A,\tau}_{\alpha,k}(t) y_{k-1} + \int_{-\tau}^{0} R^{A,\tau}_{\alpha,\alpha}(t-\tau-u) A\phi(u)\mathrm{d}u.$$

Proof. Analogously as above, applying (2.1), (2.3) on (1.4)–(1.6), we obtain

$$\begin{aligned}
&\mathcal{L}(y(t))(s) \\
&= (s^\alpha I - A\exp\{-s\tau\})^{-1} \left[\sum_{k=1}^{\lceil\alpha\rceil} s^{\alpha-k} y_{k-1} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\} A\phi(t)\mathrm{d}t\right] \\
&= \sum_{k=1}^{\lceil\alpha\rceil} \mathcal{L}(R^{A,\tau}_{\alpha,k}(t))(s) y_{k-1} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\} \mathcal{L}(R^{A,\tau}_{\alpha,\alpha}(t))(s) A\phi(t)\mathrm{d}t
\end{aligned}$$

which again concludes the proof. □

Remark 3.3. We can see that the integral terms involving the initial function ϕ are for y_{RL} and y_C identical. The difference occurs in the terms involving the local initial conditions. Although the Caputo case is more studied in the literature, in particular of order $\alpha \in (0, 1]$ (see, e.g. [1,3,6]), the Riemann-Liouville one actually appears to be structurally closer to the classical case. Indeed, $R^{A,\tau}_{\alpha,\alpha}$ seems to be playing practically the same role as the fundamental matrix solution in integer-order delay differential equations.

It might look like Theorems 3.1 and 3.2 are not that much explicit since the R-functions are defined via the inverse Laplace transform. Now we show that these functions can be actually evaluated pretty straighforwardly.

Applying the Jordan canonical form theory, we can write

$$\mathcal{L}(R^{A,\tau}_{\alpha,\beta}(t))(s) = (s^\alpha I - A\exp\{-s\tau\})^{-1} s^{\alpha-\beta} = T(s^\alpha I - \Lambda\exp\{-s\tau\})^{-1} s^{\alpha-\beta} T^{-1}\,.$$

Clearly, the matrix $(s^{\alpha}I - \Lambda\exp\{-s\tau\})^{-1}s^{\alpha-\beta}$ is block diagonal with the blocks given by upper triangular strip matrices of the form

$$(s^{\alpha}I - J_k\mathrm{e}^{-s\tau})^{-1}s^{\alpha-\beta} = \begin{pmatrix} \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau}} & \frac{\mathrm{e}^{-s\tau}s^{\alpha-\beta}}{(s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau})^2} & \cdots & \frac{\mathrm{e}^{-(r-1)s\tau}s^{\alpha-\beta}}{(s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau})^{r_k}} \\ 0 & \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau}} & \ddots & \frac{\mathrm{e}^{-(r-2)s\tau}s^{\alpha-\beta}}{(s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau})^{r_k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda_i\mathrm{e}^{-s\tau}} \end{pmatrix}, \tag{3.1}$$

where J_k $(k = 1, \dots, q)$ is the k-th block of Λ and r_k is its size. It was proven in [1] that the elements of this matrix can be expressed as

$$\frac{\exp\{-ms\tau\}s^{\alpha-\beta}}{(s^{\alpha} - \lambda\exp\{-s\tau\})^{m+1}} = \mathcal{L}(G^{\lambda,\tau,m}_{\alpha,\beta}(t))(s)$$

where

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{j=0}^{\lceil t/\tau-m-1\rceil} \binom{m+j}{j} \frac{\lambda^j(t-(m+j)\tau)^{\alpha(m+j)+\beta-1}}{\Gamma(\alpha(m+j)+\beta)}, \quad t > 0\,.$$

To summarize the previous considerations, we can write the following assertion.

Lemma 3.4. *Let $A \in \mathbb{R}^{d\times d}$, λ_i $(i = 1, \dots, n)$ be distinct eigenvalues of A and let p_i be the largest size of the Jordan block corresponding to the eigenvalue λ_i. Then the non-zero elements of matrix function $R^{A,\tau}_{\alpha,\beta}$ are linear combinations of scalar functions*

$$G^{\lambda_i,\tau,m}_{\alpha,\beta}(t), \quad m = 0, \dots, p_i - 1, \quad i = 1, \dots, n\,.$$

4. Main results

It is well known from the basic theory of the Laplace transform method that if all poles of the Laplace image of solutions (roots of the so-called characteristic equation) have negative real parts, then the zero solution of the studied equation is asymptotically stable (and their non-zero solutions tend to zero in an exponential rate). On the other hand, if there exists a pole with a positive real part, the corresponding zero solution is not stable (its absolute value tends to infinity exponentially). In the fractional case, it usually occurs a more complex situation, involving also singular points and poles with the zero real parts, which require a deeper analysis.

For our fractional problems (1.1) and (1.4), as it can be seen from the proof of Theorems 3.1 and 3.2, the characteristic equation takes the form

$$\det(s^{\alpha}I - A\exp\{-s\tau\}) = 0 \quad \text{or} \quad \prod_{i=1}^{n}(s^{\alpha} - \lambda_i\exp\{-s\tau\})^{w_i} = 0\,, \tag{4.1}$$

where λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A and w_i are the corresponding algebraic multiplicities. As we can see from (4.1) and (3.1), for further eigenvalues considerations it is sufficient to investigate the roots of the equation

$$p(s;\lambda) \equiv s^{\alpha} - \lambda\exp\{-s\tau\} = 0 \tag{4.2}$$

where λ is a complex parameter. Now, we perform a direct root analysis of (4.2). In particular, we formulate the optimal conditions on λ ensuring that (4.2) does not have any root with positive real part.

Lemma 4.1. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$, $\tau > 0$ and $\lambda \in \mathbb{C}$. Then all the roots of (4.2) have negative real parts if and only if*

$$\alpha \in (0,2)\,, \quad |\mathrm{Arg}\,(\lambda)| > \frac{\alpha\pi}{2} \quad \text{and} \quad |\lambda| < \left(\frac{|\mathrm{Arg}\,(\lambda)| - \alpha\pi/2}{\tau}\right)^{\alpha} \tag{4.3}$$

where $\mathrm{Arg}\,(\lambda) \in (-\pi, \pi]$ *is the principal argument of* λ.

Proof. The case $\lambda = 0$ is trivial since then (4.2) has only the zero solution which does not satisfy (4.3). Let $\lambda \neq 0$ and put

$$s = r\exp\{\mathrm{i}\varphi\}\,, \quad \lambda = \varrho\exp\{\mathrm{i}\psi\}$$

where $r = |s|$, $\varrho = |\lambda|$ and $\varphi, \psi \in (-\pi, \pi]$ are principal arguments of s, λ, respectively. Then we can write (4.2) for real and imaginary parts as a system of two equations in the form

$$r^{\alpha}\cos(\alpha\varphi) - \varrho\exp\{-r\tau\cos(\varphi)\}\cos(\psi - r\tau\sin(\varphi)) = 0, \tag{4.4}$$

$$r^{\alpha}\sin(\alpha\varphi) - \varrho\exp\{-r\tau\cos(\varphi)\}\sin(\psi - r\tau\sin(\varphi)) = 0. \tag{4.5}$$

Now, let us assume that (4.2) has a root with a non-negative real part, i.e. $|\varphi| \leq \pi/2$.

For $\varphi = 0$, we have $\psi = 0$ (i.e. $\lambda = \varrho$) from (4.5). Further, (4.4) implies, for r and ϱ, the relation $r^{\alpha} = \varrho\exp\{-r\tau\}$ which always allows to find an appropriate r to a given ϱ. Hence, (4.2) has a non-negative real root if and only if λ is a non-negative real.

Let $|\varphi| \in (0, \pi/2] \setminus \{\pi/\alpha\}$. Since $|\varphi| \neq \pi/\alpha$, we have $\psi - r\tau\sin(\varphi) \neq k\pi$ for any $k \in \mathbb{Z}$ and, by dividing and rearranging (4.4) and (4.5), we arrive at a new reformulation of (4.4), (4.5) in the form

$$\alpha\varphi = \psi - r\tau\sin(\varphi) + 2k\pi, \tag{4.6}$$

$$r^{\alpha} = \varrho\exp\{-r\tau\cos(\varphi)\} \tag{4.7}$$

for a suitable $k \in \mathbb{Z}$ (the replacement of $k\pi$ by $2k\pi$ is implied by positivity of r and ϱ). Further, by eliminating r from (4.6), (4.7), we get the equation for φ as

$$\left(\frac{\psi - \alpha\varphi + 2k\pi}{\tau\sin(\varphi)}\right)^{\alpha} = \varrho\exp\{(\alpha\varphi - \psi - 2k\pi)\cot(\varphi)\}\,.$$

As proven in [1] for $\alpha \in (0,1)$, the left-hand side is decreasing with respect to φ on $(0, \pi/2]$ with the lowest value at $\varphi = \pi/2$ for any k. The right-hand side is increasing with respect to φ on $(0, \pi/2]$ with the largest value at $\varphi = \pi/2$ for any k. It can be easily checked that the situation for $\alpha \geq 1$ is the same provided we put the left-hand side equal to zero for φ such that $\psi - \alpha\varphi + 2k\pi < 0$. Obviously, the existence of a root $\varphi \in (0, \pi/2]$ for at least one k is ensured if and only if

$$|\psi| \leq \frac{\alpha\pi}{2} \quad \text{or} \quad \varrho \geq \left(\frac{|\psi| - \alpha\pi/2}{\tau}\right)^{\alpha}. \tag{4.8}$$

We can see that $(4.8)_1$ is automatically satisfied for $\alpha \geq 2$, hence for $\alpha \geq 2$ there is always a root of (4.2) with a non-negative real part. We can see that, for $\alpha \in (0,2)$, (4.8) is a complement of (4.3).

So far, we have not investigated the situation $|\varphi| = \pi/\alpha \leq \pi/2$. However, it can occur only for $\alpha \geq 2$ and in that case we already know that there is always a root of (4.2) with a non-negative real part.

Summarizing the previous arguments, we can conclude the proof. □

Lemma 3.4 shows that functions of the type $G^{\lambda,\tau,m}_{\alpha,\beta}$ play, for (1.1) and (1.4), an analogous role as exponential functions for integer-order systems. Hence, it is crucial to have a good uderstanding of asymptotic behavior of $G^{\lambda,\tau,m}_{\alpha,\beta}$ and its relation to (4.2) which is provided by the following assertion which slightly extends the result presented in [1].

Lemma 4.2. *Let $\lambda \in \mathbb{C}$, $\alpha, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}$ be such that $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \geq 0$. Further, let s_i $(i = 1, 2, \dots)$ be the roots of (4.2) with ordering $\Re(s_i) \geq \Re(s_{i+1})$ (in particular, s_1 is the zero with the largest real part).*

(i) *If $\lambda = 0$, then*

$$G^{0,\tau,m}_{\alpha,\beta}(t) = \frac{(t - m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha + \beta)} h(t - m\tau).$$

(ii) *If λ is such that s_1 has negative real part, then*

$$\begin{aligned} G^{\lambda,\tau,m}_{\alpha,\beta}(t) =& \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta - \alpha)}(t + \tau)^{\beta-\alpha-1} \\ &+ \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta - 2\alpha)}(t + 2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \qquad \text{as } t \to \infty. \end{aligned}$$

(iii) *If λ is such that s_1 is purely imaginary or it has positive real part, then*

$$\begin{aligned} G^{\lambda,\tau,m}_{\alpha,\beta}(t) =& \sum_{j=0}^{m} (t - m\tau)^j (a_j \exp\{s_1(t - m\tau)\} \\ &+ b_j \exp\{s_2(t - m\tau)\}) + \begin{cases} \mathcal{O}(t^m \exp\{\Re(s_3)t\}), & \text{if } \Re(s_3) \geq 0, \\ \mathcal{O}(t^{\beta-\alpha-1}), & \text{if } \Re(s_3) < 0 \end{cases} \\ & \text{as } t \to \infty \end{aligned}$$

where a_j, b_j are suitable nonzero complex constants $(j = 0, \dots, m)$.

Proof. The assertion was proved in [1] for the case $\alpha \in (0,1)$. The generalization for $\alpha > 1$ is a tedious but direct analogue. □

Now we are in a position to formulate the main results of this paper. For the sake of lucidity, we introduce the following subset of complex numbers motivated by (4.3) as

$$\mathcal{S}_{\alpha,\tau} = \left\{ \lambda \in \mathbb{C} : |\lambda| < \left(\frac{|\mathrm{Arg}\,(\lambda)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \ |\mathrm{Arg}\,(\lambda)| > \frac{\alpha\pi}{2} \right\},$$

which we call the stability region of (1.1) and (1.4).

Theorem 4.3. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $\tau > 0$. Further, let $p_0 \in \mathbb{Z}$ be the largest size of the Jordan block corresponding to the zero eigenvalue of A, where we put $p_0 = 0$ if A has only non-zero eigenvalues.*

(i) *The zero solution of (1.1) is asymptotically stable if and only if $\alpha \in (0,2)$, all non-zero eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$ and $p_0 < 1/\alpha$.*

(ii) *The zero solution of (1.1) is stable if and only if $\alpha \in (0,2)$, all eigenvalues of A belong to $\mathrm{cl}\,(\mathcal{S}_{\alpha,\tau})$, all non-zero eigenvalues of A lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities and $p_0 \le 1/\alpha$.*

Proof. Theorem 3.1 and Lemma 3.4 imply that the solution components of (1.1) are formed as linear combinations of functions

$$G^{\lambda_i,\tau,m}_{\alpha,\alpha-k+1}(t) \quad \text{and} \quad \int_{-\tau}^{0} G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi_j(u)\mathrm{d}u\,, \tag{4.9}$$

where $k = 1,\ldots,\lceil\alpha\rceil$, λ_i $(i = 1,\ldots,n)$ are eigenvalues of A, m is a non-negative integer as specified in Lemma 3.4 and ϕ_j $(j = 1,\ldots,d)$ are components of the initial function.

Lemma 4.1 implies that all roots of (4.1) have negative real part if and only if $\alpha \in (0,2)$ and all eigenvalues belong to $\mathcal{S}_{\alpha,\tau}$. Moreover, (4.1) has at least one root with zero real part and other roots with a negative real part if and only if at least one eigenvalue lies on the boundary of $\mathcal{S}_{\alpha,\tau}$.

Thus, the asymptotic behavior of the solution can be derived from Lemma 4.2. The functions $(4.9)_1$ are described directly, we just point out that for $\lambda_i \in \mathcal{S}_{\alpha,\tau}$ the first term in the expansion cancels out due to the negative integer argument in the Gamma function, so that we obtain

$$G^{\lambda_i,\tau,m}_{\alpha,\alpha-k+1}(t) = \frac{(-1)^{m+1}(m+1)}{\lambda_i^{m+2}\Gamma(-\alpha-k+1)}(t+2\tau)^{-\alpha-k} + \mathcal{O}(t^{-2\alpha-k}) \qquad \text{as } t\to\infty.$$

Now, we investigate $(4.9)_2$. Employing the assuption $\phi \in L^1[-\tau,0]$ and Lemma 4.2, we can distinguish several cases:

Let $\alpha \in (0,2)$ and $\lambda_i \in \mathcal{S}_{\alpha,\tau}$. The second mean value theorem implies

$$\begin{aligned}\int_{-\tau}^{0} G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u &= G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t)\int_{-\tau}^{\xi}\phi(u)\mathrm{d}u \\ &= K_1(t+2\tau)^{-\alpha-1} + \mathcal{O}(t^{-2\alpha-1}) \qquad \text{as } t\to\infty\,,\end{aligned}$$

where $K_1 \in \mathbb{R}$ is non-zero and $\xi \in (-\tau,0]$.

Now, let $\lambda_i = 0$. By the same approach we arrive at

$$\int_{-\tau}^{0} G^{0,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u = K_2(t-m\tau)^{(m+1)\alpha-1}\,,$$

where $K_2 \in \mathbb{R}$ is non-zero. This expression vanishes for $t\to\infty$, if and only if $m+1 = p_0 < 1/\alpha$.

The cases for $\lambda_i \in \partial\mathcal{S}_{\alpha,\tau}\setminus\{0\}$ and $\lambda_i \notin \mathrm{cl}\,(\mathcal{S}_{\alpha,\tau})$ can be handled similarly. We arrive at the conclusion that $(4.9)_2$ is bounded, when the non-zero eigenvalue lying on the boundary of stability region has the same algebraic and geometric multiplicity. Otherwise the absolute value of $(4.9)_2$ increases polynomially (when the eigenvalue lies on the boundary) or exponentially. □

Theorem 4.4. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $\tau > 0$. Further, let $p_0 \in \mathbb{Z}$ be the largest size of the Jordan block corresponding to the zero eigenvalue of A, where we put $p_0 = 0$ if A has only non-zero eigenvalues.*

(i) *The zero solution of (1.4) is asymptotically stable if and only if $\alpha \in (0,2)$ and all eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$.*

(ii) *The zero solution of (1.4) is stable if and only if $\alpha \in (0,2]$, all eigenvalues of A belong to $\operatorname{cl}(\mathcal{S}_{\alpha,\tau})$, all non-zero eigenvalues of A lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities and $p_0 \le 2 - \lceil\alpha\rceil$.*

Proof. The idea of the proof is equivalent to that one of Theorem 4.3. In particular, the solution components of (1.4) are given by linear combinations of

$$G_{\alpha,k}^{\lambda_i,\tau,m}(t) \quad \text{and} \quad \int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\phi_j(u)\mathrm{d}u\,, \tag{4.10}$$

where $k = 1,\dots,\lceil\alpha\rceil$, λ_i $(i = 1,\dots,n)$ are eigenvalues of A, m is a non-negative integer as specified in Lemma 3.4 and ϕ_j $(j = 1,\dots,d)$ is a component of the initial function. Thus, we see that $(4.10)_2$ is the same as $(4.9)_2$ while $(4.10)_1$ differs with respect to $(4.9)_1$ due to the change of index. This causes only a different decay rate for $\lambda_i \in \operatorname{cl}(\mathcal{S}_{\alpha,\tau})$.

Overall, there is only one difference in stability behavior which occurs for $\lambda_i = 0$ when we have

$$G_{\alpha,k}^{0,\tau,m}(t) = \frac{(t-m\tau)^{m\alpha+k-1}}{\Gamma(m\alpha+k)}.$$

We can see that this function never tends to zero with $t \to \infty$ and it is bounded if and only if $m\alpha + k - 1 = 0$ which means $\lceil\alpha\rceil = 1$ (i.e. $k = 1$) and $p_0 = 1$ (i.e. $m = 0$). ☐

Remark 4.5. (i) Theorems 4.3 and 4.4 show that $\mathcal{S}_{\alpha,\tau}$ is the stability region for delayed fractional differential systems for Riemann-Liouville and Caputo derivative, i.e. for (1.1) and (1.4), respectively. Figure 1 represents the situation for $\alpha \in (0,1)$ when the stability region includes also points with positive real part. We can see in Figure 2 how the region is transformed for $\alpha \in (1,2)$, and it is apparent how the stability region vanishes for $\alpha \to 2$. Also, for $\tau \to 0$, $\mathcal{S}_{\alpha,\tau}$ tends to the stability region known from theory of fractional differential equations without delay (see, e.g. [7,9]).

(ii) From the stability viewpoint, the only difference between (1.1) and (1.4) occurs if there is a zero eigenvalue and the order of derivatives is less than 1. In this case, the zero solution to (1.1) can be asymptotically stable, stable or unstable, depending on the particular value of α and multiplicities of the zero eigenvalue. The zero solution of (1.4) is stable if algebraic and geometric multiplicities of the zero eigenvalue are equal, otherwise it is unstable (i.e. it does not depend on the particular value of α).

The proof technique used for Theorems 4.3 and 4.4 actually reveals more than the stability properties. Due to its constructive nature we can actually derive also the asymptotic behavior of the solutions to (1.1) and (1.4). We summarize the comparisons of the two cases in the following assertions dealing with the asymptotic

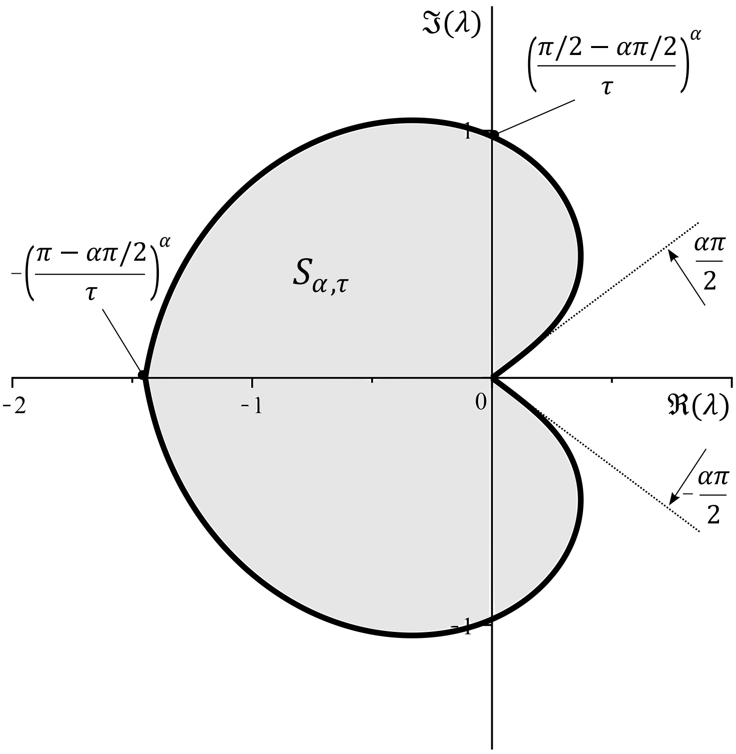

Figure 1. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.4$ and $\tau = 1$.

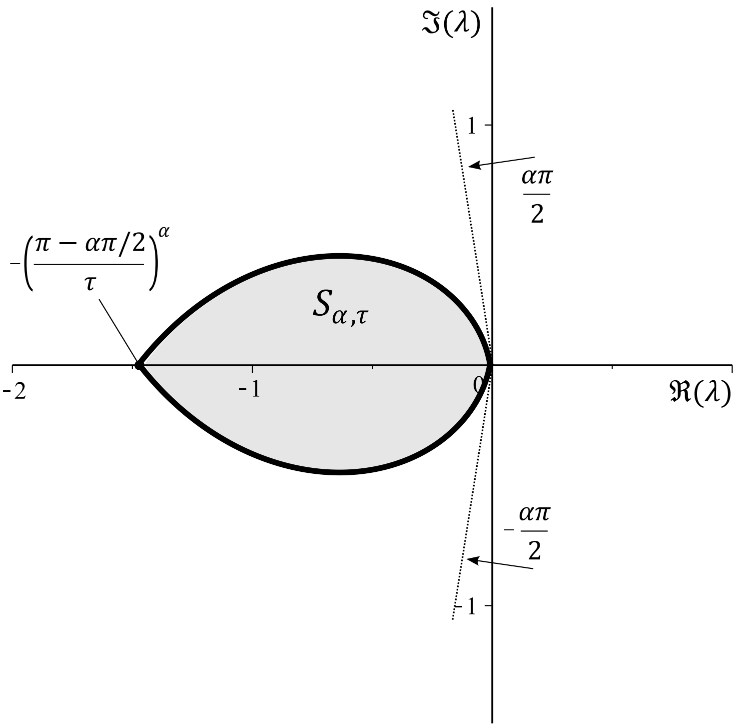

Figure 2. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 1.1$ and $\tau = 1$.

equivalence (denoted by the symbol $\sim$) relationships for norms of solutions (we use the symbol $\|\cdot\|$ for Euclidean norm in $\mathbb{R}^d$).

Theorem 4.6. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0,2)$, $\tau > 0$ and let all the eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$. Further, we denote by y_{RL} and y_C the solutions of (1.1)–(1.3) and (1.4)–(1.6), respectively. Then it holds*

$$\|y_{RL}(t)\| \sim t^{-\alpha-1} \quad \text{and} \quad \|y_C(t)\| \sim t^{\lceil\alpha\rceil-\alpha-1} \qquad \text{as } t \to \infty \tag{4.11}$$

for almost all choices of initial conditions. If y_{RL} and y_C do not follow (4.11), then their norms tend to zero with a faster decay rate.

Proof. Theorems 3.1 and 3.2 indicate some particular choices of initial conditions, e.g. $y_0 = 0$, which can remove the dominating terms from y_{RL} and y_C and therefore affect the decay rate. The particular asymptotic properties are then implied by Lemma 4.2. □

Theorem 4.7. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0,2)$ and $\tau > 0$. Let A has the zero eigenvalue and denote p_0 the size of the largest Jordan block corresponding to this zero eigenvalue. Let all non-zero eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$. Further, we denote y_{RL} and y_C the solutions of (1.1)–(1.3) and (1.4)–(1.6), respectively. Then it holds*

$$\|y_{RL}(t)\| \sim t^{p_0\alpha-1} \quad \text{and} \quad \|y_C(t)\| \sim t^{(p_0-1)\alpha+\lceil\alpha\rceil-1} \qquad \text{as } t \to \infty \tag{4.12}$$

for almost all choices of initial conditions. If y_{RL} and y_C do not follow (4.12), then their norms are even smaller for t large enough.

Proof. The idea of the proof is analogous to the previous case. □

Remark 4.8. (i) We can observe an interesting distinction between the way how the asymptotic behavior of y_{RL} and y_C depends on α. While in the Riemann–Liouville case we see the algebraic decay rate depending directly on α, in the Caputo case the decay rate is driven by the decimal part of α, i.e. by the difference $\lceil\alpha\rceil - \alpha$. Indeed, if we consider for example $\alpha_1 = 0.4$ and $\alpha_2 = 1.4$, then the solutions of (1.4) follow essentially the same asymptotic relations, while the Riemann–Liouville ones do not.

(ii) We can employ a similar analysis also in the cases that are not covered by Theorems 4.6 and 4.7, i.e. when there is a non-zero eigenvalue on the boundary or outside the closure of the stability region. We note that if there is a non-zero eigenvalue lying outside the closure of the stability region, the norms of non-zero solutions increase exponentially for both (1.1) and (1.4).

(iii) We point out that the asymptotic results obtained for the delayed fractional differential systems actually mirror the well-known results for fractional differential systems without a delay.

5. Conclusions

We have summarized and extended the results on qualitative behavior of solutions of delayed fractional differential systems (1.1) and (1.4) of arbitrary order.

We have shown that the stability of the zero solution occurs only if the order of derivatives is less than 2. Further, we have derived the precise description of the stability region which is for both (1.1) and (1.4) identical. The only difference regarding the stability occurs when the system matrix A has a zero eigenvalue. Then we observe the asymptotic stability property for (1.1) only if $\alpha < 1$ and the maximum size of the Jordan block corresponding to the zero eigenvalue being less than $1/\alpha$. In the Caputo case (1.4), the asymptotic stability does not appear and the zero solution is stable (but not asymptotically stable) only if $\alpha < 1$ and algebraic multiplicity of the zero eigenvalue being equal to the geometric one.

The asymptotic behavior displays more diversity. If the system matrix A has all eigenvalues lying in the stability region, i.e. the zero solutions of both (1.1) and (1.4) are asymptotically stable, we can generally say that the solutions of (1.1) go to zero as $t \to \infty$ faster that solutions of (1.4). Moreover, unlike the Riemann-Liouville case, the decay rate of solutions to (1.4) does not depend on the value α itself, but on its decimal part only.

The area of qualitative analysis of fractional differential equations with a time delay, especially with higher-order derivative, provides a lot of open problems. Our research may serve as one of the prerequisites to studies of more complex systems, such as $\mathbf{D}_0^\alpha y(t) = ay(t) + by(t-\tau)$ or its vector analogues.

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Appendix D

Paper on Lambert function and one-term FDDE [15] (FCAA, 2023)

The topic of one-term FDDS seemed, in principle, mostly complete to us, at least regarding stable solutions. We revisited it only recently in a re-union of the author's trio, which was primarily active before 2016. The inspiration for [15] (co-authors: J. Čermák, L. Nechvátal; my author's share 33 %) came from discussions nearly nine years ago about a possible fractional generalization of the classical Lambert function technique known from the stability analysis of ordinary delay differential equations.

As it turned out, this generalization is not only possible but, in some cases, easier than other known approaches. We managed to re-derive fully explicit criteria that had not been reached by this technique before, contributing to a better understanding of unbounded solutions for higher-order FDDS. As a by-product, we created an "asymptotic map" for unbounded solutions, showing their large-time exponential modulus growth and frequency of oscillations based on the location of system matrix eigenvalue.

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ORIGINAL PAPER

The Lambert function method in qualitative analysis of fractional delay differential equations

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Abstract

We discuss an analytical method for qualitative investigations of linear fractional delay differential equations. This method originates from the Lambert function technique that is traditionally used in stability analysis of ordinary delay differential equations. Contrary to the existing results based on such a technique, we show that the method can result into fully explicit stability criteria for a linear fractional delay differential equation, supported by a precise description of its asymptotics. As a by-product of our investigations, we also state alternate proofs of some classical assertions that are given in a more lucid form compared to the existing proofs.

Keywords Fractional delay differential equation (primary) · Lambert function · Stability · Asymptotic behavior

Mathematics Subject Classification (Primary) 34K37 · 33E30 · 33E12 · 34K20 · 34K25

1 Introduction

The paper discusses an analytical method for qualitative investigations of fractional delay differential equations (FDDEs). These equations are currently very intensively studied due to their importance in various application areas, with a special emphasis to control theory. Indeed, presence of both the time lag as well as non-integer derivative

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order as control or tunning parameters in studied models provides a very efficient tool for various control processes such as stabilization or destabilization of the particular solutions of these models (for a pioneering work in this direction we refer to [17]).

Systematic investigations of FDDEs were initiated in the paper [9]. Here, stability properties of

$$\mathrm{D}^{\alpha}x(t) = \lambda x(t-\tau), \tag{1}$$

where $\alpha, \tau \in \mathbb{R}^+$, $\lambda \in \mathbb{R}$ and D^{α} is a fractional differential operator, were analyzed using the fact that (1) is asymptotically stable (i.e., any its solution is eventually tending to zero) if and only if all the roots of the characteristic equation

$$s^{\alpha} - \lambda \exp(-s\tau) = 0 \tag{2}$$

have negative real parts. To explore such a location of characteristic roots with respect to the imaginary axis, the Lambert function technique was utilized. The essence of the method consists in a representation formula for characteristic roots in terms of appropriate branches of this multi-valued function (for some precisions concerning the correct use of the Lambert function technique in stability analysis of (1), we refer also to [12]). A certain general disadvantage of this approach consists in its (seeming) disability to provide stability criteria in an explicit form depending on entry parameters only (i.e., on α, λ and τ in the case of (1)).

As the other papers on stability and asymptotic properties of (1) followed, the Lambert function method was replaced by some alternate classical tools of stability investigations (such as D–partition method or τ-decomposition method) modified to the fractional case. Using these approaches, effective and non-improvable stability conditions for (1), supported by some asymptotic bounds, were derived in [16] (the case $\lambda \in \mathbb{R}$, $0 < \alpha < 1$), [6] (the case $\lambda \in \mathbb{C}$, $0 < \alpha < 1$), and partially also in [7] (the case $\lambda \in \mathbb{C}$, $\alpha > 0$). Some of the mentioned results can be extended also to the case of a two–term FDDE

$$\mathrm{D}^{\alpha}x(t) = \mu x(t) + \lambda x(t-\tau). \tag{3}$$

In this respect, we refer to [2, 5, 15] (the case $\mu, \lambda \in \mathbb{R}$, $0 < \alpha < 1$) and [8] (the case $\mu, \lambda \in \mathbb{R}$, $1 < \alpha < 2$). Following the integer-order case (see, e.g., [1]), (1) and (3) may serve as test equations for numerical analysis of FDDEs. From this point of view, it is very important to describe their basic qualitative properties in the strongest possible form. Then, when analyzing appropriate numerical schemes applied to these test equations, the ability to keep the key qualitative properties of the underlying exact equations is of basic importance. For some other recent advances in qualitative theory of FDDEs, we refer, e.g., to [3, 10, 11, 18–20, 23].

Following the above outlines, the aim of this paper is twofold. First, we deepen the existing knowledge on some qualitative properties of (1) with the Caputo fractional derivative. Second, perhaps a more important aspect of the paper consists in the way how we aim to do it. We come back to the Lambert function method used in [9] and show that this approach can offer more than formulae depending on the use of supporting software packages. In fact, this technique can result into actually effective stability and asymptotic criteria.

The paper is organized as follows. Section 2 recalls some existing findings on (1) and essentials of the Lambert function theory. In Sect. 3, we explore the Lambert function method in details. In particular, we give an alternate proof of the classical assertion saying that the characteristic root generated by the principal branch of the Lambert function has the largest real part, and formulate a criterion that enables to localize values of the principal branch in the complex plane. Section 4 presents applications of these results to (1). Here, we extend the existing stability criteria for (1) to arbitrary (positive) real values of α, and formulate sharp asymptotic estimates for the solutions of (1). Some final remarks in Sect. 5 conclude the paper.

2 Basic mathematical background

In this section, we summarize some known facts relevant to our next investigations. First, we recall a close relationship between stability and asymptotic properties of (1), and distribution of the characteristic roots of (2). Then, we recall some basics of the Lambert function and its use in stability analysis of FDDEs.

It was shown in [7] that any solution x of (1) with the Caputo fractional derivative (and a generally complex λ) can be written using the Mittag-Leffler type function

$$G^{\lambda,\tau}_{\alpha,\beta}(t) = \sum_{j=0}^{\lceil t/\tau\rceil - 1} \frac{\lambda^j (t - j\tau)^{\alpha j+\beta-1}}{\Gamma(\alpha j + \beta)}, \qquad \alpha, \beta > 0, \tag{4}$$

where $\lceil\cdot\rceil$ denotes the upper integer part. More precisely, if ϕ is a continuous initial (complex-valued) function on $[-\tau, 0]$, $\phi_0 = \phi(0)$ and ϕ_j, $j = 1, \ldots, \lceil\alpha\rceil - 1$, are (complex) constants (considered when $\alpha > 1$), then

$$x(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j G^{\lambda,\tau}_{\alpha,j+1}(t) + \lambda \int_{-\tau}^{0} G^{\lambda,\tau}_{\alpha,\alpha}(t - \tau - \xi)\phi(\xi)\,\mathrm{d}\xi \tag{5}$$

is the solution of (1) satisfying $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$, and $\lim_{t\to 0^+} x^{(j)}(t) = \phi_j$, $j = 1, \ldots, \lceil\alpha\rceil - 1$.

Based on some asymptotic results on (4), the solution (5) can be rewritten by the use of the characteristic roots having non-negative real parts. We recall that (2) admits countably many roots, and only a finite number of them is lying right to any line $\Re(s) = p$, $p \in \mathbb{R}$ (throughout the paper, the symbol $\Re(z)$ and $\Im(z)$ stands for the real and imaginary part of $z \in \mathbb{C}$, respectively). If we denote by S the set of all roots of (2) having non-negative real parts (note that S must be a finite set), then, for a non-integer α, (5) can be rewritten as

$$x(t) = \sum_{s\in S} c_s \exp(st) + \mathcal{O}(t^{j-\alpha}) \quad \text{as } t \to \infty \tag{6}$$

where c_s are complex coefficients depending on $\alpha, \tau, \lambda, \phi$, and $j \in \{-1, 0, \ldots, \lceil\alpha\rceil - 1\}$ (the particular value of j depends on limit behavior of ϕ at $t = 0$). Notice that $j - \alpha < 0$, i.e., the function $t^{j-\alpha}$ always tends to zero.

By (6), the roots of (2) play an essential role in qualitative behavior of the solutions of (1). Following the classical integer-order pattern, the authors in [9] used the following chain of steps

$$s^\alpha \exp(s\tau) = \lambda \quad \rightarrow \quad s \exp\left(\frac{\tau}{\alpha} s\right) = \lambda^{\frac{1}{\alpha}} \quad \rightarrow \quad \frac{\tau}{\alpha} s \exp\left(\frac{\tau}{\alpha} s\right) = \frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}} \tag{7}$$

to express the roots of (2) via the Lambert function introduced as the solution of

$$W(z)\exp(W(z)) = z, \quad z \in \mathbb{C}. \tag{8}$$

Before we recall the root formula for (2) based on this special function, some of its basic properties might be collected. The Lambert function is a multi-valued function (except at $z = 0$) with infinitely many (single-valued) branches W_k, $k \in \mathbb{Z}$. Neither of them can be expressed in terms of elementary functions. In particular, W_0 is called a principal branch. For any $z \in \mathbb{C}$, $\Im(W_0(z))$ is between $-\pi$ and π. The other branches are numbered so that $\Im(W_k(z))$ is between $(2k-2)\pi$ and $(2k+1)\pi$ while $\Im(W_{-k}(z))$ is between $-(2k+1)\pi$ and $-(2k-2)\pi$ for any $z \in \mathbb{C}$ and $k = 1, 2, \ldots$. More precisely, the ranges of $W_{\pm k}$ and $W_{\pm(k+1)}$, $k = 0, 1, \ldots$, are separated by the curves

$$\{w = x + \mathrm{i}\, y \in \mathbb{C} : x = -y\cot(y),\ 2k\pi < |y| < (2k+1)\pi\}$$

and the ranges of W_1 and W_{-1} are separated by the half-line

$$\{w = x + \mathrm{i}\, y \in \mathbb{C} : -\infty < x \le -1,\ y = 0\}.$$

These separating curves correspond to the branch cuts in the z-plane defined as

$$\{z = \xi + \mathrm{i}\,\eta \in \mathbb{C} : -\infty < \xi \le -\exp(-1),\ \eta = 0\}$$

in the case of W_0, and

$$\{z = \xi + \mathrm{i}\,\eta \in \mathbb{C} : -\infty < \xi \le 0,\ \eta = 0\}$$

in the case of W_k, $k \neq 0$. Conventionally, the branch cut (having the argument π in the z-plane) is mapped by W_k on its upper boundary in the w-plane. Only the branches W_0 and W_{-1} take on real values for a real $z \in [-\exp(-1), \infty)$ and a real $z \in [-\exp(-1), 0)$, respectively. Further details on the Lambert function (including some historical remarks) can be found in [4], for other comments, see also [13] and [22].

Now, following (7), all the roots of (2) can be expressed in the form

$$s_k = \frac{\alpha}{\tau} W_k\left(\frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}}\right), \quad k \in \mathbb{Z}. \tag{9}$$

By (6), a crucial role in analysis of (1) is played by the rightmost characteristic root (i.e., the root of (2) with the largest real part). The following classical assertion says that this root is just s_0.

Lemma 1 *Let $z \in \mathbb{C}$. Then $W_0(z)$ has the largest real part $\Re(W_0(z))$ among all the other real parts $\Re(W_k(z))$, $k \in \mathbb{Z}$.*

The original proof of Lemma 1 is pretty long (see [22]). As a by-product of our next procedures, we are going to present an alternate (and more simple) way how to prove this assertion.

Remark 1 As pointed out in [12], the expression (9) is not quite correct for some complex values of λ. More precisely, (7) contains taking the $1/\alpha$-power which means that the roots given by (9) are identical to those of (2) only in the case

$$|\mathrm{Arg}(\lambda)| \leq \alpha\pi$$

(we recall that $-\pi < \mathrm{Arg}(\cdot) \leq \pi$). This inequality is satisfied trivially when $\alpha \geq 1$ but makes a restriction when $0 < \alpha < 1$. In other words, if $|\mathrm{Arg}(\lambda)| > \alpha\pi$, then the representation (9) can produce some superfluous roots that are actually not the true roots of (2). As an example, we can consider, e.g., the case $\lambda = -1, \alpha = 1/2$ and $\tau = 1$ when (2) has the rightmost root $s_0 \approx -0.4172 - \mathrm{i}\,2.2651$ (i.e., (1) is asymptotically stable) while (9) produces $s_0 \approx 0.4263 > 0$. On this account, we discuss qualitative properties of (1) for $\alpha > 1$. Comments to the case $0 < \alpha < 1$ are provided in the final section.

3 Some advances on the Lambert *W* function

This section contains several key results on the Lambert function which proved to be useful in qualitative investigations of (1). To obtain an actually effective and strong asymptotic description of the solutions of (1), we need to effectively localize the position of the rightmost characteristic root in the complex plane. More precisely, by (6) and (9), we need to derive effective expressions of the real and imaginary parts of $W_0(z)$ in terms of z. Thus, keeping in mind intended stability and asymptotic analysis of (1), we can pose the following problems: *For given $p \in \mathbb{R}$ and $z \in \mathbb{C}$, is it possible to characterize the properties $\Re(W_0(z)) < p$ and $\Re(W_0(z)) = p$ directly in terms of z and p, i.e., without an evaluation of the principal branch of the Lambert function? Further, for given $q \in \mathbb{R}$ and $z \in \mathbb{C}$, is it possible to similarly elaborate on the properties $|\Im(W_0(z))| > q$ and $|\Im(W_0(z))| = q$?* The following result yields an affirmative answer to these questions.

Theorem 1 *Let $p, q \in \mathbb{R}$, $p > -1$, $0 < q < \pi$, and $z \in \mathbb{C}$, $z \neq 0$. Then*
(i) *$\Re(W_0(z)) < p$ if and only if either $|z| < p\exp(p)$ or*

$$|z| \geq |p|\exp(p) \quad \textit{and} \quad \arccos\left(\frac{p\exp(p)}{|z|}\right) + \frac{\sqrt{|z|^2 - p^2\exp(2p)}}{\exp(p)} < |\mathrm{Arg}(z)|\,; \tag{10}$$

(ii) $\Re(W_0(z)) = p$ *if and only if*

$$|z| \geq |p|\exp(p) \quad \textit{and} \quad \arccos\left(\frac{p\exp(p)}{|z|}\right) + \frac{\sqrt{|z|^2 - p^2\exp(2p)}}{\exp(p)} = |\mathrm{Arg}(z)|\,; \tag{11}$$

(iii) $|\Im(W_0(z))| > q$ *if and only if*

$$|\mathrm{Arg}(z)| > q \quad \textit{and} \quad \frac{q}{\sin(|\mathrm{Arg}(z)| - q)}\exp\big(q\cot(|\mathrm{Arg}(z)| - q)\big) < |z|; \tag{12}$$

(iv) $|\Im(W_0(z))| = q$ *if and only if*

$$|\mathrm{Arg}(z)| > q \quad \textit{and} \quad \frac{q}{\sin(|\mathrm{Arg}(z)| - q)}\exp\big(q\cot(|\mathrm{Arg}(z)| - q)\big) = |z|. \tag{13}$$

Proof (i) We write $z = |z|\exp(\mathrm{i}\,\mathrm{Arg}(z))$ and put $x_k = \Re(W_k(z))$, $y_k = \Im(W_k(z))$ where W_k, $k \in \mathbb{Z}$ are particular branches of the Lambert function. Substitution into (8) yields

$$\exp(x_k)(x_k\cos(y_k) - y_k\sin(y_k)) = |z|\cos(\mathrm{Arg}(z)), \tag{14}$$
$$\exp(x_k)(x_k\sin(y_k) + y_k\cos(y_k)) = |z|\sin(\mathrm{Arg}(z)). \tag{15}$$

If we solve (14)–(15) with respect to unknowns $x_k\exp(x_k)$ and $y_k\exp(x_k)$, then

$$x_k\exp(x_k) = |z|\cos(\mathrm{Arg}(z) - y_k), \tag{16}$$
$$y_k\exp(x_k) = |z|\sin(\mathrm{Arg}(z) - y_k). \tag{17}$$

To show that $x_0 = \Re(W_0)(z)) < p$ whenever $|z| < p\exp(p)$, we consider (16) implying

$$x_0\exp(x_0) \leq |z| < p\exp(p).$$

Then the monotony property of the function $g(p) = p\exp(p)$ on $(-1, \infty)$ actually implies $x_0 < p$.

Now we assume that $|z| \geq |p|\exp(p)$. Squaring and adding (16) and (17) we get

$$|z|^2 = ((x_k)^2 + (y_k)^2)\exp(2x_k),$$

i.e.,

$$|y_k| = \frac{\sqrt{|z|^2 - (x_k)^2\exp(2x_k)}}{\exp(x_k)}. \tag{18}$$

For the principal branch, it holds $x_0 \geq -y_0 \cot(y_0)$, $|y_0| < \pi$, i.e., $x_0 \sin(y_0) + y_0 \cos(y_0) \geq 0$ whenever $y_0 \geq 0$. Multiplying this by $\exp(x_0)$ and using (15), one gets

$$|z| \sin(\mathrm{Arg}(z)) = \exp(x_0)(x_0 \sin(y_0) + y_0 \cos(y_0)) \geq 0$$

which implies $\mathrm{Arg}(z) \geq 0$ for $y_0 \geq 0$. If $y_0 < 0$, the same argumentation leads to $\mathrm{Arg}(z) \leq 0$, hence $\mathrm{Arg}(z) y_0 \geq 0$, i.e., $|\mathrm{Arg}(z) - y_0| \leq \pi$. Then (16) with $k = 0$ is equivalent to

$$\arccos(x_0 \exp(x_0)/|z|) = |\mathrm{Arg}(z) - y_0| . \tag{19}$$

Moreover, sign analysis of (17) with respect to $\mathrm{Arg}(z) y_0 \geq 0$ yields $|\mathrm{Arg}(z)| \geq |y_0|$, i.e.,

$$|\mathrm{Arg}(z) - y_0| = |\mathrm{Arg}(z)| - |y_0|. \tag{20}$$

Then, using (18), (19) and (20), we are able to set up an implicit dependence between $x_0 = \Re(W_0(z))$ and z in the form $f(x_0, z) = 0$ where f is defined via

$$f(p, z) = \arccos\left(\frac{p \exp(p)}{|z|}\right) - |\mathrm{Arg}(z)| + \frac{\sqrt{|z|^2 - p^2 \exp(2p)}}{\exp(p)}$$

for all $p > -1$ and $z \in \mathbb{C}$ such that $|p| \exp(p) \leq |z|$. Let z be fixed. Then

$$\frac{\mathrm{d}f}{\mathrm{d}p}(p, z) = -\frac{(2p+1)\exp(3p) + |z|^2 \exp(p)}{\exp(2p)\sqrt{|z|^2 - p^2 \exp(2p)}} \leq -\frac{(p+1)^2 \exp(p)}{\sqrt{|z|^2 - p^2 \exp(2p)}} \leq 0,$$

hence, f is decreasing in p if $|p| \exp(p) \leq |z|$. Therefore,

$$f(p, z) < f(x_0, z) = 0$$

whenever

$$p > x_0 = \Re(W_0(z)) \quad \text{and} \quad |p| \exp(p) \leq |z|.$$

(ii) The property follows directly from the proof of (i) using the fact that $f(p, z) = 0$ if and only if $p = x_0$ due to monotony of f with respect to p.

(iii) Since W_0 is symmetric in the sense $W_0(\bar{z}) = \overline{W_0(z)}$ for all $z \in \mathbb{C}$ except those lying on the branch cut along the negative real axis between $-\infty$ and $-\exp(-1)$, it suffices to assume the case $y_0 = \Im(W_0(z)) > q > 0$. We have already observed that $\mathrm{Arg}(z) \geq y_0$. In addition, a stronger property holds, namely $\mathrm{Arg}(z) > y_0$. Indeed, possible equality $\mathrm{Arg}(z) = y_0$ implies $y_0 = 0$ (due to (17)) which contradicts the assumption $y_0 > q > 0$. Hence, it must be $0 < \mathrm{Arg}(z) - y_0 < \pi$ as well as $0 < \mathrm{Arg}(z) - q < \pi$. We divide (16) by (17) and put $k = 0$ to get

$$x_0 = y_0 \cot(\mathrm{Arg}(z) - y_0). \tag{21}$$

Taking the logarithm of (17) with $k = 0$, we also have

$$x_0 = \ln\big(|z|\sin(\mathrm{Arg}(z) - y_0)\big) - \ln(y_0). \tag{22}$$

Combining (21) and (22), we arrive at

$$\ln\big(|z|\sin(\mathrm{Arg}(z) - y_0)\big) - \ln(y_0) - y_0\cot(\mathrm{Arg}(z) - y_0) = 0$$

representing again an implicit dependence, now between $\Im(W_0(z))$ and z. If we denote

$$h(q,z) = \ln\big(|z|\sin(\mathrm{Arg}(z) - q)\big) - \ln(q) - q\cot(\mathrm{Arg}(z) - q),$$

then we have

$$\frac{\mathrm{d}h}{\mathrm{d}q}(q,z) = \frac{-q\sin(2(\mathrm{Arg}(z) - q)) - \sin^2(\mathrm{Arg}(z) - q) - q^2}{q\sin^2(\mathrm{Arg}(z) - q)}.$$

While the denominator is positive, the numerator

$$N(q,z) = -q\sin(2(\mathrm{Arg}(z) - q)) - \sin^2(\mathrm{Arg}(z) - q) - q^2$$

is negative for each $0 \le q \le \mathrm{Arg}(z)$. Indeed, we have

$$\frac{\mathrm{d}N}{\mathrm{d}q}(q,z) = 2q(\cos^2(\mathrm{Arg}(z) - q) - 1) \le 0$$

which implies that $N(\cdot, z)$ is non-increasing and, together with $N(0,z) = -\sin^2(\mathrm{Arg}(z)) < 0$, negative on $[0, \mathrm{Arg}(z)]$. Consequently, $h(\cdot, z)$ is decreasing and therefore $h(q,z) > h(y_0, z) = 0$ whenever $0 < q < y_0 < \mathrm{Arg}(z)$. Taking into account the above mentioned symmetry, we arrive (after some elementary algebra) at (12).

(iv) The required property is again a consequence of monotony of the function h from the previous part. □

Remark 2 (a) The properties (ii) and (iv) of Theorem 1 provide a new tool for evaluations of the principal branch of the Lambert function. Let $z \neq 0$ be a fixed complex number. Then the left-hand side of (11) is decreasing for all $p \in [a, W_0(|z|)]$ ($a = -1$ if $|z| \ge \exp(-1)$ and $a = W_0(-|z|)$ if $|z| < \exp(-1)$) from π to the zero value. Hence, (11) has a unique root p^* lying in this interval, and this root equals just $\Re(W_0(z))$. Similarly, the left-hand side of (13) is increasing for all $q \in (0, \mathrm{Arg}(z))$ from the zero value to infinity, i.e., (13) admits a unique positive root q^* which is just $\Im(W_0(z))$.

To illustrate this evaluation technique, we compute $W_0(z)$ for $z = \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$. Then $|z| = 1$, $\mathrm{Arg}(z) = \pi/3$ and the standard Newton method returns $\Re(z) = p^* \approx 0.4843$ in 5 iterations with the initial value $p_0 = 0.5$ and the stopping criterion taken as $|p_{k+1} - p_k| \le 10^{-16}$. The same method gives $\Im(z) = q^* \approx 0.3808$ in 7 iterations with the initial value $q_0 = 0.5$ and the same precision as in the case of the real part.

In fact, the value $p^* + \mathrm{i}\, q^*$ matches the value produced by the MATLAB command `lambertw(1/2+sqrt(3)/2*1i)` to all the 15 digits behind the decimal point. Standardly, the Newton or Halley method is applied directly to the equation $w\exp(w) - z = 0$ using the complex arithmetic. MATLAB employs the latter method with some advanced guess of the starting point. For computing the values of the Lambert function with arbitrary precision, we refer to the recent paper [14].

(b) Using a different approach, the property (i) of Theorem 1 was also discussed in [21].

In the sequel, we clarify ordering of the real as well as imaginary parts of the particular branches of the Lambert function. This ordering may be useful in a deeper asymptotic analysis of (1), and, moreover, results into an alternate proof of Lemma 1.

Following the proof of Theorem 1, we introduce the functions

$$G_z(x, y) = x\sin(y) + y\cos(y) - |z|\sin(\mathrm{Arg}(z))\exp(-x) \quad \text{and}$$
$$f_z(x) = \sqrt{|z|^2\exp(-2x) - x^2}.$$

In view of (15) and (18), the couples (x_k, y_k), where $x_k = \Re(W_k(z))$, $y_k = \Im(W_k(z))$, have to meet the relations $G_z(x, y) = 0$ and $y = \pm f_z(x)$, respectively.

The following assertion specifies ordering of imaginary parts of the branches of the Lambert function.

Lemma 2 *Let $z \in \mathbb{C} \setminus \{0\}$. Then $\Im(W_k(z)) \le \Im(W_{k+1}(z))$ for all $k \in \mathbb{Z}$. In fact, all the inequalities are strict with the only exception: If $z \in [-\exp(-1), 0)$, then we have $\Im(W_{-1}(z)) = \Im(W_0(z)) = 0$.*

Proof For the sake of formal simplicity, we identify complex numbers $w = x + \mathrm{i}\, y$ with couples $(x, y) \in \mathbb{R}^2$. First, let $z \in \mathbb{C} \setminus \{0\}$ be such that $0 \le \mathrm{Arg}(z) \le \pi$ and define sets S_j^z, $j \in \mathbb{Z}$, as

$$\begin{aligned}
S_j^z &= \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ (2j-1)\pi < y < (2j+1)\pi\} && \text{for } j = 1, 2, \ldots;\\
S_j^z &= \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ 0 \le y < \pi\} && \text{for } j = 0;\\
S_j^z &= \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ -2\pi < y \le 0\} && \text{for } j = -1;\\
S_j^z &= \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ 2j\pi < y < (2j+2)\pi\} && \text{for } j = -2, -3, \ldots
\end{aligned}$$

(note that the equation $G_z(x, y) = 0$ has no solution for $y = (2j-1)\pi$, $j = 1, 2, \ldots$, and for $y = 2j\pi$, $j = -1, -2, \ldots$). We wish to show that S_j^z is a part of the range of W_k just when $j = k$.

Let $k \ge 1$ be arbitrary. Then, by the definition of W_k (see also Sect. 2),

$$(2k-2)\pi < y_k < (2k+1)\pi. \tag{23}$$

Let j be such that $(x_k, y_k) \in S_j^z$. We distinguish the following cases with respect to j.

If $j > k$, then $y_k > (2k+1)\pi$ which contradicts (23). Let $j = k$. The ranges of W_k and W_{k+1}, $k = 0, 1, \dots$, are separated by the curve

$$\gamma_k = \{(u, v) \in \mathbb{R}^2 : u = -v\cot(v),\ 2k\pi < v < (2k+1)\pi\}$$

(see Sect. 2). The equation $G_z(x, y) = 0$, $(2k-1)\pi < y < (2k+1)\pi$, is equivalent to

$$x = -y\cot(y) + |z|\sin(\mathrm{Arg}(z))\exp(-x) \tag{24}$$

provided $y \neq 2k\pi$. To estimate the y-coordinate of a point $(x, y) \in S_k^z$, we put $u = x$ in γ_k.

First, let $2k\pi < y < (2k+1)\pi$. Then any point (x, y) of S_k^z together with the corresponding point (x, v) of γ_k have to fulfill the formula

$$y\cot(y) - v\cot(v) = \frac{|z|\sin(\mathrm{Arg}(z))}{\sin(y)}\exp(-x) \tag{25}$$

due to $x = -v\cot(v)$ and (24). Since $0 \le \mathrm{Arg}(z) \le \pi$, the right-hand side of (25) is non-negative, hence, we have $y\cot(y) \ge v\cot(v)$ implying $y \le v$. In other words, any point $(x, y) \in S_k^z$ is located below or on the curve γ_k separating W_k and W_{k+1}. Second, let $(2k-1)\pi < y \le 2k\pi$. Then the points $(x, y) \in S_k^z$ lie below γ_k trivially (note that the equation $G_z(x, y) = 0$ has a unique solution x for $y = 2k\pi$, $0 < \mathrm{Arg}(z) < \pi$, and has no solution for $y = 2k\pi$, $\mathrm{Arg}(z) = 0$ or $\mathrm{Arg}(z) = \pi$). On the other hand, any point of S_k^z lies above the upper bound $(2k-1)\pi$ of γ_{k-1}, hence, we have proven that S_k^z is contained in the range of W_k.

Finally, if $j < k$, then we can similarly verify that any point of S_j^z is already located below or on γ_{k-1}. Thus, to summarize the previous observations, S_j^z is contained in the range of W_k $(k = 1, 2, \dots)$ just when $j = k$; otherwise, S_j^z and the range of W_k are disjoint. Consequently, $(2k-1)\pi < y_k < (2k+1)\pi$, hence, $y_k < y_{k+1}$ for all $k = 1, 2, \dots$.

The same line of arguments can be used in the case $k \le -1$ to obtain $2k\pi < y_k < (2k+2)\pi$, and, in the remaining case $k = 0$, to obtain $0 \le y_0 < \pi$ (and therefore, $y_0 < y_1$). In this respect, a real z is mapped by W_0 to γ_0 (hence, $y_0 > 0$) if $z < -\exp(-1)$, and is mapped by W_0 to reals if $z \ge -\exp(-1)$ (hence, $y_0 = 0$). Also W_{-1} takes real values just when $-\exp(-1) \le z < 0$ (hence, $y_{-1} = y_0 = 0$).

The procedure can be analogously applied to the case $-\pi < \mathrm{Arg}(z) < 0$ (definition of the sets S_j^z now differ by shifting the particular y-domains vertically down by π). □

Lemma 2 is useful also for ordering of the real parts of the branches of the Lambert function. In particular, it enables to prove the classical assertion of Lemma 1 in a more lucid way compared to the existing proof techniques.

Proof of Lemma 1 We recall that the couple (x_k, y_k), $k \in \mathbb{Z}$, satisfies $|y| = f_z(x)$, see the text preceding Lemma 2. Put $\zeta_1 = W_{-1}(-|z|)$, $\zeta_2 = W_0(-|z|)$, $\zeta_3 = W_0(|z|)$.

Then, we can easily observe that f_z is defined on $(-\infty, \zeta_3]$ and $(-\infty, \zeta_1] \cup [\zeta_2, \zeta_3]$ if $|z| \geq \exp(-1)$ and $0 < |z| < \exp(-1)$, respectively. Moreover, f_z has a (unique) root $\zeta_3 > 0$ if $|z| > \exp(-1)$, a couple of roots $\zeta_1 = \zeta_2 = -1$ and $\zeta_3 > 1$ if $|z| = \exp(-1)$, and a triple of roots $\zeta_1 < -1, -1 < \zeta_2 < 0, \zeta_3 > 0$ if $0 < |z| < \exp(-1)$. Otherwise, f_z is positive at all other points of its domain.

Further, we have

$$f_z'(x) = \frac{-|z|^2 \exp(-2x) - x}{\sqrt{|z|^2 \exp(-2x) - x^2}}.$$

If $|z| \geq \frac{\sqrt{2}}{2} \exp(-1/2)$, then f_z' is negative, hence f_z is decreasing on $(-\infty, \zeta_3)$. If $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, then f_z has a local minimum at $\zeta_4 = \frac{1}{2} W_{-1}(-2|z|^2)$ and a local maximum at $\zeta_5 = \frac{1}{2} W_0(-2|z|^2)$ (note that $\zeta_4 < \zeta_5$). Consequently, f_z is decreasing on $(-\infty, \zeta_4)$, increasing on (ζ_4, ζ_5) and again decreasing on (ζ_5, ζ_3).

If $|z| = \exp(-1)$, then f_z is decreasing on $(-\infty, -1)$, increasing on $(-1, \zeta_5)$ and decreasing on (ζ_5, ζ_3). Finally, if $\exp(-1) > |z| > 0$, then f_z is decreasing on $(-\infty, \zeta_1)$ increasing on (ζ_2, ζ_5) and decreasing on (ζ_5, ζ_3).

Thus, f_z is decreasing on its domain up to "a small part" which is, however, lying within the range of W_0 (we again identify $w = x + \mathrm{i}\, y$ with $(x, y) \in \mathbb{R}^2$). Indeed, if $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, we have to show that the graph of f_z between the points $[\zeta_4, f_z(\zeta_4)]$ and $[\zeta_5, f_z(\zeta_5)]$ is contained in the range of W_0. Since

$$\zeta_4^2 + f_z^2(\zeta_4) = -\frac{1}{2} W_{-1}(-2|z|) < 1 \quad \text{and} \quad \zeta_5^2 + f_z^2(\zeta_5) = -\frac{1}{2} W_0(-2|z|^2) < \frac{1}{2}$$

for any $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, both the endpoints of the graph belong to the range of W_0 (note that the open unit disk is a part of the W_0 range). Also, if $\zeta_4 \leq x \leq \zeta_5$, then $x^2 + f_z^2 = |z| \exp(-2x)$ is decreasing in x, hence, we have $|z| \exp(-2x) < 1$ for any $\zeta_4 \leq x \leq \zeta_5$ and any $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$ meaning that the graph of the increasing part of f_z is again lying in the range of W_0.

Similarly, we can show that, for $\exp(-1) \geq |z| > 0$, the graph of f_z between the relevant points is contained in the range of W_0 as well.

Collecting the above monotony properties together with Lemma 2, we can conclude that $x_{\pm(k+1)} < x_{\pm k}$ for any $k = 1, 2, \ldots$. If $k = 0$, we have $x_1 < x_0$ but $x_{-1} \leq x_0$ because of $W_0(z) = \overline{W_{-1}(z)}$ for any $z \in \mathbb{R}, z \leq -\exp(-1)$. □

Remark 3 We have actually proven monotony of the real parts of $W_k(z)$ with respect to k which is a slightly stronger result than that stated in Lemma 1.

4 Applications towards qualitative properties of (1) with the Caputo derivative

In this section, we apply our previous observations on the principal branch of the Lambert function to describe important qualitative properties of (1), including their

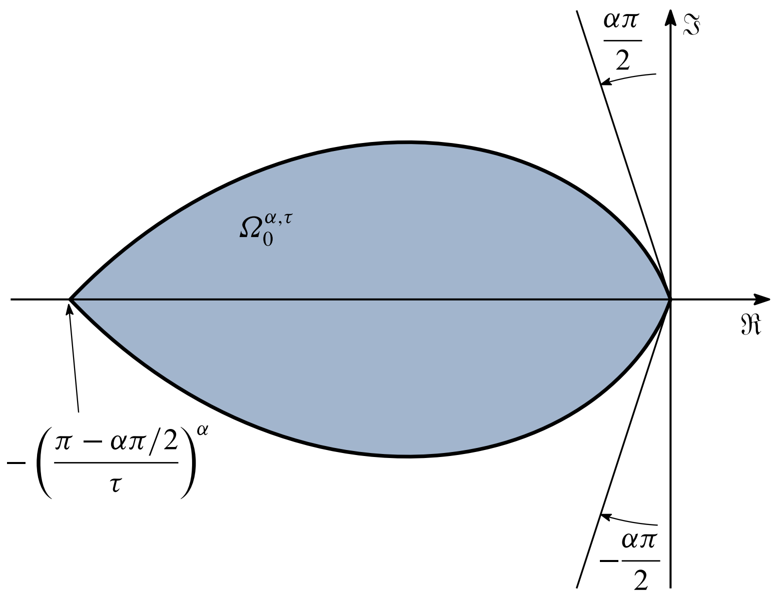

Fig. 1 The set $\Omega_0^{\alpha,\tau}$ as the stability region for (1) (the figure corresponds to $\alpha = 1.2$)

dependence on the parameter λ. We remind that here we restrict ourselves to the case $\alpha > 1$ due to the reason mentioned in Remark 1.

We start with a basic stability criterion for (1). We introduce a (parametric) set $\Omega_0^{\alpha,\tau}$ given as

$$\Omega_0^{\alpha,\tau} = \left\{ z \in \mathbb{C} : |z| < \left(\frac{|\mathrm{Arg}(z)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \quad |\mathrm{Arg}(z)| > \frac{\alpha\pi}{2} \right\}, \tag{26}$$

see Fig. 1. This set already appeared in [6, Thm. 2] as the asymptotic stability region for (1) with $0 < \alpha < 1$. As indicated in [7], the D-subdivision method (combined with some tools of fractional calculus) used in [6] is extendable also to the case $\alpha > 1$. In the following assertion, we confirm validity of this stability result for $\alpha > 1$. Contrary to the existing techniques, we are able to prove this result (as a consequence of Theorem 1(i)) in an almost elementary way.

Theorem 2 *Let $\alpha > 1$, $\tau > 0$ and $\lambda \in \mathbb{C}$. Then (1) is asymptotically stable if and only if $\lambda \in \Omega_0^{\alpha,\tau}$.*

Proof By (9) and Lemma 1, we need to analyze $\Re(s_0) = \frac{\alpha}{\tau}\Re\big(W_0\big(\frac{\tau}{\alpha}\lambda^{1/\alpha}\big)\big) < 0$. Using (10) with $z = \tau\lambda^{1/\alpha}/\alpha$ and $p = 0$, this inequality can be converted into

$$\left|\frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}}\right| + \frac{\pi}{2} < \left|\mathrm{Arg}(\lambda^{\frac{1}{\alpha}})\right|,$$

i.e.,

$$\frac{\tau}{\alpha}|\lambda|^{\frac{1}{\alpha}} < \left|\mathrm{Arg}(\lambda^{\frac{1}{\alpha}})\right| - \frac{\pi}{2}.$$

Taking into account $\left|\mathrm{Arg}(\lambda^{1/\alpha})\right| = \frac{1}{\alpha}\left|\mathrm{Arg}(\lambda)\right|$, this is equivalent to the condition defining $\Omega_0^{\alpha,\tau}$ in (26). □

Remark 4 If $\lambda \in \partial\Omega_0^{\alpha,\tau}$, then (1) is stable but not asymptotically stable. If $\lambda \notin \mathrm{cl}\Omega_0^{\alpha,\tau}$, then (1) is unstable. Obviously, $\Omega_0^{\alpha,\tau}$ becomes empty for any $\alpha \geq 2$, hence, (1) cannot be asymptotically stable for any complex λ whenever $\alpha \geq 2$.

Theorem 1 can be used in a more subtle way to bring a deeper insight into behavior of (1). More precisely, relations (10)–(13) enable to reveal a relationship between the position of the rightmost characteristic root s_0 and the value of λ. Then, by (6), such a relationship easily results into a precise asymptotic description of the solutions of (1) in the unstable case ($\Re(s_0) > 0$). Indeed, while in the asymptotically stable case ($\Re(s_0) < 0$) the decay rate of solutions is algebraic and independent of the particular value of s_0, in the unstable case ($\Re(s_0) > 0$), the growth rate is exponential and governed by the real part of the rightmost characteristic root s_0 (it is well known that this is true also for integer values of α). In addition, the imaginary part of s_0 is related to the frequency characteristics describing an oscillatory behavior of (1).

Thus, for given $u, v \geq 0$, we need to find a region of all complex λ such that the rightmost characteristic root s_0 is lying on or left to the line $u + \mathrm{i}\,\omega$, $\omega \in \mathbb{R}$, and on or above (below) the line $\omega + \mathrm{i}\,v$ (the line $\omega - \mathrm{i}\,v$), $\omega \in \mathbb{R}$. On this account, similarly as in the proof of Theorem 2, we put $z = \tau\lambda^{1/\alpha}/\alpha$, $p = \tau u/\alpha$, $q = \tau v/\alpha$ and consider $u \geq 0$, $\alpha\pi/\tau > v > 0$. Then, Theorem 1 immediately implies that

(i) $\Re(s_0) \leq u$ if and only if either

$$|\lambda| < u^\alpha \exp(\tau u) \tag{27}$$

or

$$|\lambda| \geq u^\alpha \exp(\tau u) \quad \text{and} \quad \alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)} \\ \leq |\mathrm{Arg}(\lambda)|\,; \tag{28}$$

(ii) $|\Im(s_0)| \geq v$ if and only if

$$|\mathrm{Arg}(\lambda)| > \tau v/\alpha \quad \text{and} \quad \frac{v^\alpha}{\sin^\alpha\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)} \\ \times \exp\big(\tau v\cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\big) \leq |\lambda|. \tag{29}$$

Thus, if we introduce the set $\overline{\Omega}_u^{\alpha,\tau}$ as a set of all $\lambda \in \mathbb{C}$ such that either (27) or (28) holds, and $\overline{\Psi}_v^{\alpha,\tau}$ as a set of all $\lambda \in \mathbb{C}$ such that (29) holds (we put $\overline{\Psi}_v^{\alpha,\tau} = \emptyset$ for $\alpha\pi/\tau \leq v < \pi$ and $\overline{\Psi}_v^{\alpha,\tau} = \mathbb{C}$ for $v = 0$), then we can rewrite our previous observations into the following assertion:

Lemma 3 *Let $\alpha > 1$, $\tau > 0$, $u \geq 0$, $\pi > v \geq 0$, $\lambda \in \mathbb{C}$ and let s_0 be given by (9). Then*

(i) $\Re(s_0) \le u$ *if and only if* $\lambda \in \overline{\Omega}_u^{\alpha,\tau}$;
(ii) $|\Im(s_0)| \ge v$ *if and only if* $\lambda \in \overline{\Psi}_v^{\alpha,\tau}$ ($\overline{\Psi}_v^{\alpha,\tau}$ *is non-empty whenever* $0 \le v < \alpha\pi/\tau$).

Remark 5 (a) The set $\overline{\Omega}_u^{\alpha,\tau}$ contains the origin ($\lambda = 0$) which is excluded by Theorem 1. However, admitting $\lambda = 0$, we have the only characteristic root $s_0 = 0$ of (2), hence, Lemma 3 remains true.

(b) It is easy to check that $\Omega_0^{\alpha,\tau}$ introduced in (26) coincides with int $\overline{\Omega}_0^{\alpha,\tau}$.

The part (i) of Lemma 3 immediately implies

Corollary 1 *Let* $u \ge 0$ *be fixed. Then* $\overline{\Omega}_u^{\alpha,\tau}$ *is the set of all* $\lambda \in \mathbb{C}$ *such that* $x(t) = \mathcal{O}(\exp(ut))$ *as* $t \to \infty$ *for any solution* x *of* (1).

To obtain an actually effective (and non-improvable) asymptotic result for the solutions of (1), we have to look at the problem inversely. More precisely, for a given complex $\lambda \notin \Omega_0^{\alpha,\tau}$, we need to find (non-negative) real values u_0, v_0 such that the rightmost root s_0 of (2) satisfies $\Re(s_0) = u_0$, $|\Im(s_0)| = v_0$.

The way how to do it easily follows from Theorems 1(ii) and 1(iv), respectively, taking into account the above introduced substitutions $z = \tau\lambda^{1/\alpha}/\alpha$, $p = \tau u/\alpha$, $q = \tau v/\alpha$. Then the corresponding relations (11) and (13) become

$$|\lambda| \ge u^\alpha \exp(\tau u) \quad \text{and} \quad \alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)} = |\mathrm{Arg}(\lambda)| \tag{30}$$

and

$$|\mathrm{Arg}(\lambda)| > \tau v/\alpha \quad \text{and} \quad \frac{v^\alpha}{\sin^\alpha\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)} \exp\big(\tau v \cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\big) = |\lambda|, \tag{31}$$

respectively (see also (28) and (29)). Thus, the unique solution of $(30)_2$ defines the value u_0, while the unique positive solution of $(31)_2$ defines the value v_0 (provided $\mathrm{Arg}(\lambda) > 0$). If $\mathrm{Arg}(\lambda) = 0$, then s_0 is real, and we set $v_0 = 0$.

Notice that $(30)_2$, defining the relation between the modulus and argument of λ that is explicit with respect to the argument, forms the boundary of $\overline{\Omega}_u^{\alpha,\tau}$. Its left-hand side, considered as a function of $|\lambda|$, is continuous, increasing and unbounded on $[u^\alpha \exp(\tau u), \infty)$, and its graph in the complex plane creates a Jordan curve symmetric with respect to the real axis, see Fig. 2.

Similarly, $(31)_2$ provides the relation between the modulus and argument of λ that is explicit with respect to the modulus. The equality $(31)_2$ is the boundary of $\overline{\Psi}_v^{\alpha,\tau}$, and its left-hand side, as a function of $|\mathrm{Arg}(\lambda)|$, is continuous on $(\tau v/\alpha, \pi]$ and unbounded in a right neighborhood of the point $\tau v/\alpha$ (for $|\mathrm{Arg}(\lambda)| = \pi$, it takes a value on the negative real axis). This implies that $\overline{\Psi}_v^{\alpha,\tau}$ is unbounded for any $0 < v < \alpha\pi/\tau$ and its boundary splits the complex plane into two parts, see Fig. 2.

Now we are in a position to formulate a complete asymptotic description for the solutions of (1).

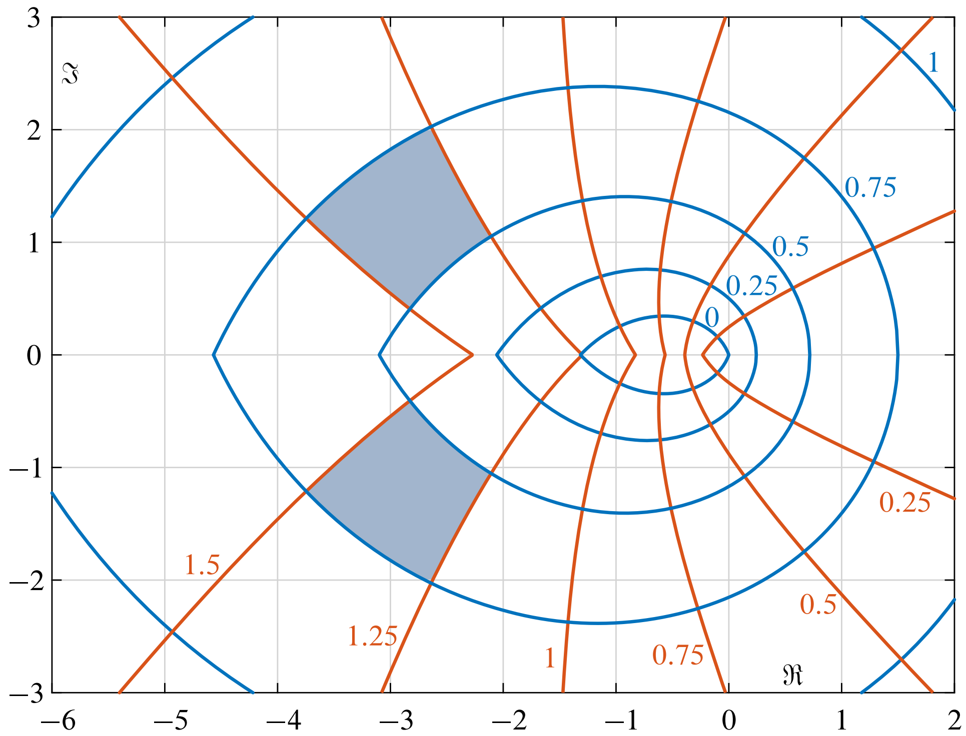

Fig. 2 The figure depicts the boundaries of the sets $\overline{\Omega}_u^{\alpha,\tau}$ (blue) and $\overline{\Psi}_v^{\alpha,\tau}$ (orange) for several values of u and v, respectively (the scenario corresponds to $\alpha = 1.2$ and $\tau = 1$). The particular blue curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_0 of (2) satisfies $\Re(s_0) = u$, and the particular orange curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_0 of (2) satisfies $|\Im(s_0)| = v$. As an example, the blueish curvilinear rectangles then represent the set of all $\lambda \in \mathbb{C}$ such that $0.5 < \Re(s_0) < 0.75$ and $1.25 < |\Im(s_0)| < 1.5$

Theorem 3 *Let* $\alpha > 1$, $\tau > 0$ *and* $\lambda \in \mathbb{C}$.

(i) *If* $\lambda \in \Omega_0^{\alpha,\tau}$, *then, for any solution* x *of* (1),

$$x(t) = \mathcal{O}(t^{1-\alpha}) \quad \textit{as} \quad t \to \infty.$$

Moreover the algebraic decay order $1 - \alpha$ *cannot be improved;*

(ii) *If* $\lambda \notin \Omega_0^{\alpha,\tau}$, *then, for any solution* x *of* (1),

$$x(t) = \exp(u_0 t)(c \exp(\mathrm{i}\, v_0 t) + o(1)) \quad \textit{as} \quad t \to \infty$$

where c *is a complex constant,* $u_0 \geq 0$ *is the unique solution of* $(30)_2$, $v_0 > 0$ *is the unique solution of* $(31)_2$ *if* $|\mathrm{Arg}(\lambda)| > 0$, *and* $v_0 = 0$ *if* $\mathrm{Arg}(\lambda) = 0$.

Proof (i) The property is a direct consequence of (6) as the set S is empty.

(ii) Let $s_0 = u_0 + \mathrm{i}\, v_0$ be the rightmost characteristic root, and S_0 be the set of the remaining characteristic roots s with a non-negative real part (we remind that $\Re(s) < \Re(s_0)$ for any $s \in S_0$). Then, if α is a non-integer, we can write (6) as

$$\begin{aligned} x(t) &= \exp(u_0 t)\Big(c \exp(\mathrm{i}\, v_0 t) + \sum_{s \in S_0} c_s \exp((s - u_0)t)\Big) + \mathcal{O}(t^{1-\alpha}) \\ &= \exp(u_0 t)\big(c \exp(\mathrm{i}\, v_0 t) + o(1) + \mathcal{O}(o(1))\big) = \exp(u_0 t)\big(c \exp(\mathrm{i}\, v_0 t) + o(1)\big) \end{aligned}$$

as $\quad t \to \infty$,

where $c = c_{s_0}$ is the complex constant from (6) corresponding to the rightmost characteristic root s_0. If α is an integer, then the dominating role of s_0 in asymptotic behavior of (1) is well known. In this case, the assertion of (ii) holds as well. □

Remark 6 (a) The asymptotic formula from Theorem 3(ii) immediately implies

$$x(t) = \mathcal{O}(\exp(u_0 t)) \quad \text{as} \quad t \to \infty \tag{32}$$

for any solution x of (1), and the constant u_0 is non-improvable. Moreover, for large t, the roots of the real and imaginary parts of x tend to the roots of $\cos(v_0 t)$ and $\sin(v_0 t)$, respectively. In both the cases, the distance between the subsequent roots tends to π/v_0. These properties are illustrated by Example 1.

(b) The asymptotic behavior of (1) significantly depends on stability of (1). In particular, the exponential terms in (6) are vanishing in the asymptotically stable case $\Re(s_0) < 0$. However, the situation changes in the limit case $\alpha = 1$ when, in accordance with the first-order theory, the rightmost characteristic root s_0 determines an exponential decay rate of the solutions also in the asymptotically stable case. Since the above argumentation can be extended to this problem as well, our results provide a contribution also to the corresponding classical first-order theory.

Example 1 Let $\alpha = 1.2$, $\tau = 1$, and consider (1) along with the initial conditions $\phi(t) = 1$ $(-1 \le t \le 0)$, $\phi_0 = \phi(0) = 1$, and $\phi_1 = \lim_{t \to 0+} x'(t) = 0$. We compare the corresponding (numerical) solutions of (1) for two distinct values of λ, namely $\lambda_1 = -2 + \mathrm{i}$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$. As indicated by Fig. 2, both the values λ_1, λ_2 lie in the instability region. In particular, the real parts u_0 of the corresponding rightmost roots are approximately 0.4721 and 0.4917, and their imaginary parts v_0 are 1.2321 and 1.5844, respectively.

The real parts of the solutions of (1) with two above specified sets of entries, along with the growth-rate functions $\exp(u_0 t)$, are depicted in Figs. 3 and 4. The graphs suggest that the modulus of constant c introduced in Theorem 3(ii) is less than one for $\lambda = \lambda_1$, and greater than one for $\lambda = \lambda_2$.

To illustrate behavior of the solutions x in better detail, Figs. 5 and 6 depict the ratio $\Re(x(t))/\exp(u_0 t)$ for λ_1 and λ_2, respectively. The resulting functions are bounded, but do not tend to zero which is a consequence of non-improvability of the constant u_0 in (32).

As mentioned in Remark 6(a), the distance between the subsequent roots of $\Re(x(t))$ tends to π/v_0. Figs. 7 and 8 illustrate this fact. We can see that while in the case of λ_1 the convergence is rather fast and the distance seems to be somewhat stabilized around the seventh root, in the case of λ_2, the stabilization occurs around the hundredth root.

5 Concluding remarks

The aim of the paper was to develop the Lambert function theory, and then apply the obtained results in qualitative investigations of (1). Using this approach, we were able

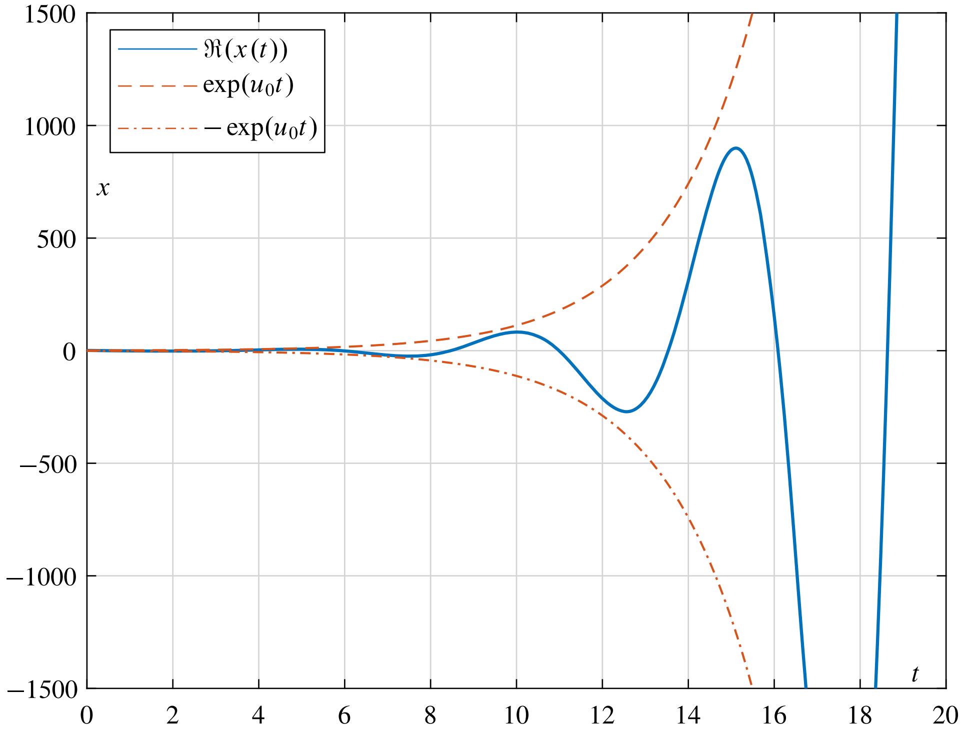

Fig. 3 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2 + \mathrm{i}$, along with the corresponding growth-rate functions $\pm\exp(0.4721t)$

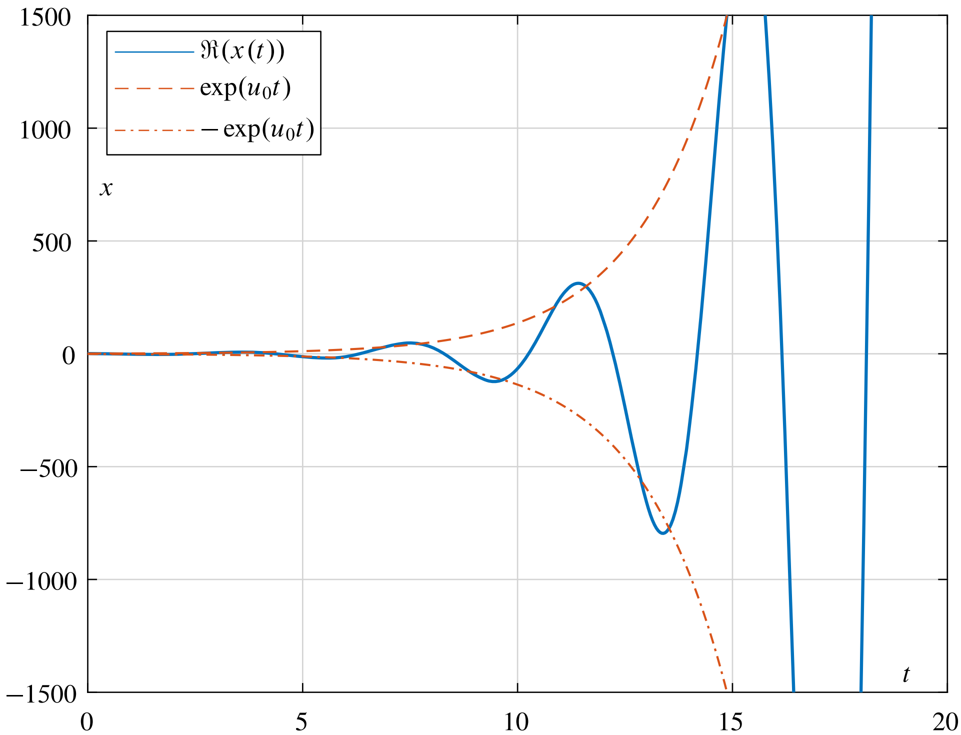

Fig. 4 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$, along with the corresponding growth-rate functions $\pm\exp(0.4917t)$

to formulate a precise asymptotic description of the solutions of (1). Particularly, in addition to an algebraic decay rate of the solutions in the stable case (described in some earlier papers), we could observe an exponential growth of the solutions in the

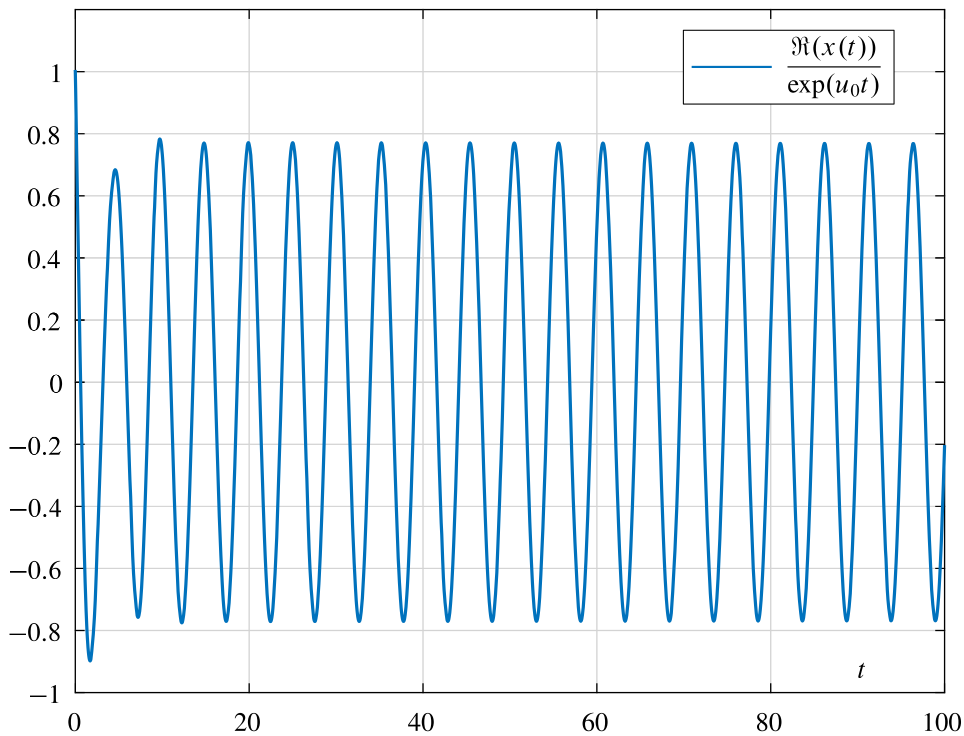

Fig. 5 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2 + \mathrm{i}$, divided by its growth-rate function $\exp(0.4721t)$

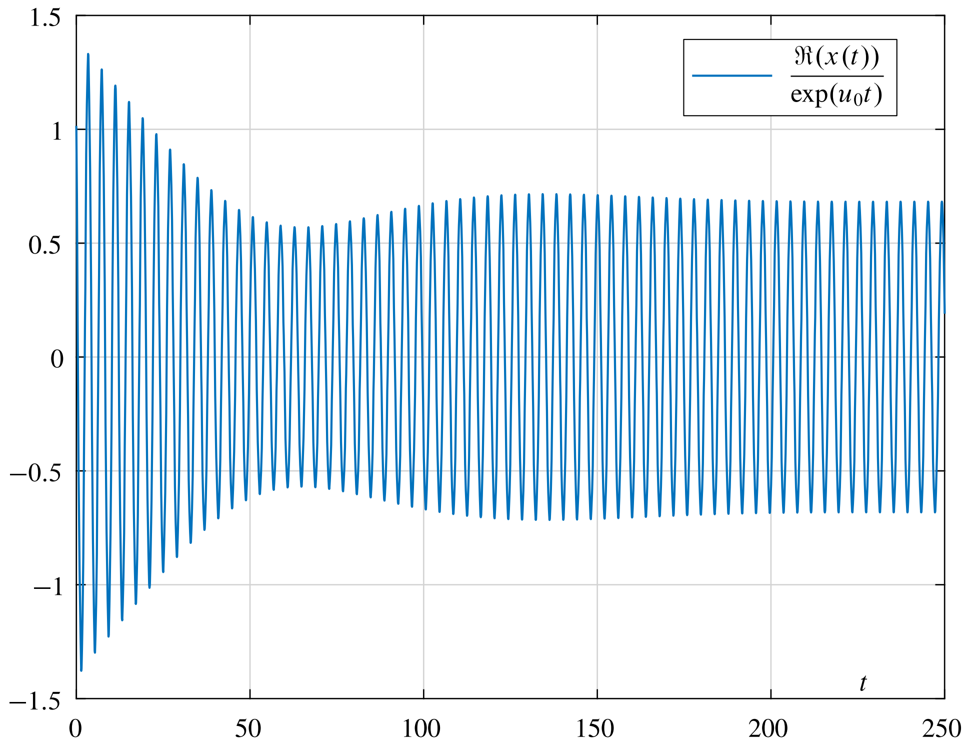

Fig. 6 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$, divided by its growth-rate function $\exp(0.4917t)$

unstable case; the rate of this growth was determined as a (unique) real root of an auxiliary transcendental equation.

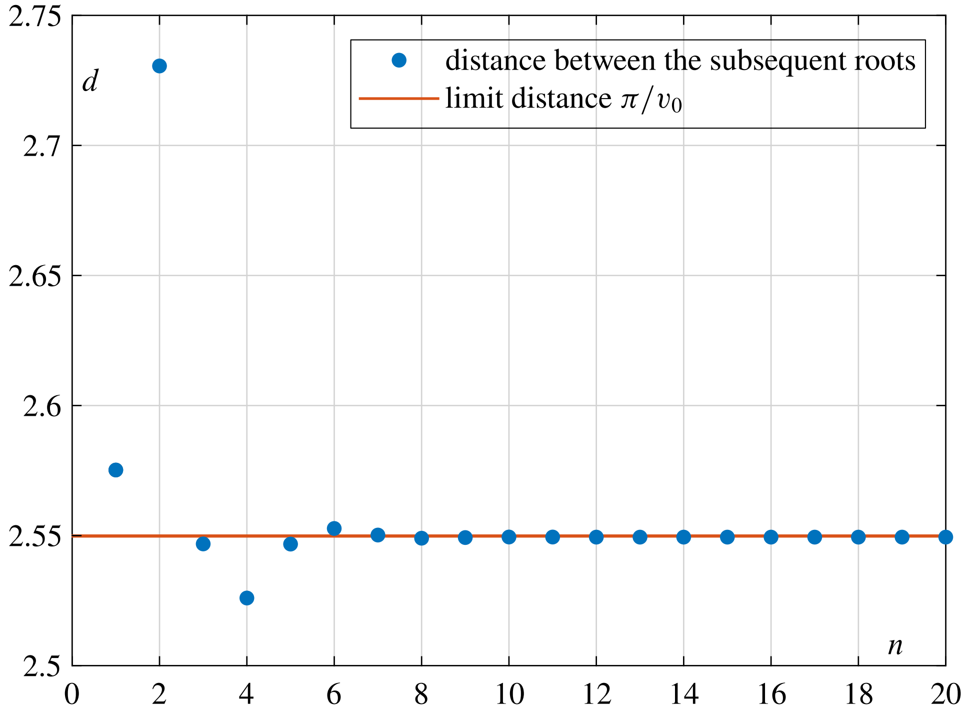

Fig. 7 The distance between the subsequent roots of $\Re(x(t))$ for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2+\mathrm{i}$ is tending to $\pi/1.2321$

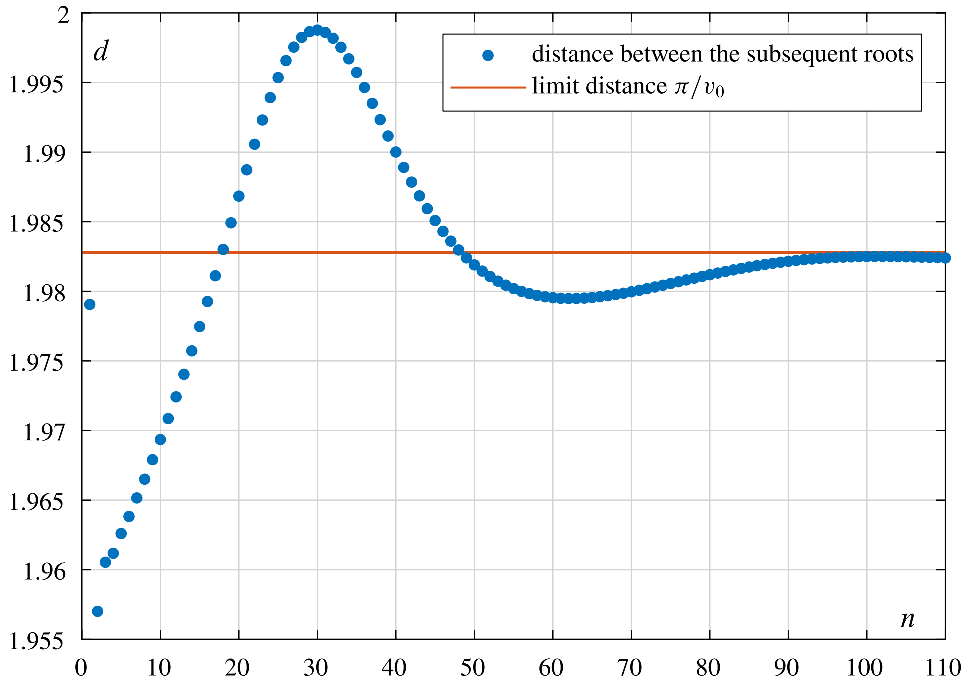

Fig. 8 The distance between the subsequent roots of $\Re(x(t))$ for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$ is tending to $\pi/1.5844$

However, the impact of the presented results is not limited to the theory of FDDEs only. Our approach offers an alternate way how to prove (and also strengthen) some classical assertions of the Lambert function theory. Moreover, to the best of our knowledge, the derived asymptotic formulae are new also in the first-order case. Here, contrary to the fractional case, our results can be applied also in the stable case where a (non-improvable) rate of exponential decay of the solutions can be determined.

Since we have formulated our results for (1) with a complex coefficient λ, their extension to the vector case is nearly straightforward provided the eigenvalues of a (real) system matrix are simple. Regarding eigenvalues with higher multiplicities, some additional argumentation seems to be necessary. Based on related cases discussed in earlier papers, one can expect a slight modification of the solutions growth, but no impact on the asymptotic frequency.

Our final remark concerns the case $0 < \alpha < 1$ not involved among the assumptions of the assertions of Sect. 4. The procedure of computing the characteristic roots uses the law of exponents which is, in general, not valid for complex numbers. Thus, some superfluous roots of the characteristic equation may appear if $0 < \alpha < 1$ (as illustrated via a counterexample in Remark 1). In this case, our stability and asymptotic formulae remain basically true, but we cannot confirm their strictness. In particular, we cannot claim that the above described rate of exponential growth of solutions is non-improvable. Nevertheless, we conjecture that a more thorough analysis of the corresponding branches of a complex power can overcome this problem, and thus achieve the strict asymptotic results for all $\alpha > 0$. Such an analysis provides another possible topic for the next research.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix E

Paper on lower-order two-term FDDE [13] (AMC, 2017)

Fairly soon after entering the field of fractional delay equations, we began to deal with linear equations involving fractional derivative, delayed term and also an undelayed one. The addition of the undelayed term brings significant technical challenges, even in scalar case, which we first addressed in [13] (co-authors: J. Čermák, Z. Došlá; my author's share 45 %).

We derived explicit necessary and sufficient conditions for asymptotic stability, including asymptotic formulas for solutions (including algebraic decay rate towards zero). To achieve this, we further developed our inverse Laplace transform technique and introduced a broad family of functions containing exponentials, Mittag-Leffler functions and generalized delay exponential of Mittag-Leffler type as special cases.

Although the stability region in space of equation's coefficients appears simple, it comprises a region of delay-independent stability and region where the stability boundary depends on the delay. This work marked my first direct encounter with the phenomenon of stability switching, where the stability property changes with increasing delay. In this case, it meant just a one-time loss of stability, but in subsequent research, much richer situations awaited us.

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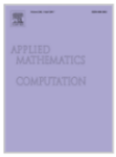

Fractional differential equations with a constant delay: Stability and asymptotics of solutions

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Due to licensing restrictions, the full text of [13] is not included in the publicly accessible part of this habilitation thesis.

Appendix F

Paper on lower-order complex two-term FDDE [11] (CNSNS, 2019)

The paper [11] (co-author: J. Čermák; my author's share 50 %) was a reaction on our growing interest in the phenomenon of stability switches. We decided to study a planar FDDS with three real entries, which can be transformed into a fractional generalization of the first-order delay differential equation with an imaginary coefficient by an undelayed term. This type of classical system is known to switch stability on and off with increasing time delay.

In this paper, we conducted a detailed analysis of stability switching, including conditions for its appearance, number, and exact calculations of stability switches. We discovered rich behaviour not present in the original first-order equation, such as different switching patterns starting from instability or stability for small delays and the presence of a delay-independent stability region.

The biggest challenge was the derivation of explicit form of the stability boundary, which is no longer given by a continuous parametric curve but by a union of segments from infinitely many parametric curves. Additionally, we explored how the stability boundary transitions with increasing order of the derivative, from a union of transcendental curves (for derivative orders less than one) to the known shape for first-order equations.

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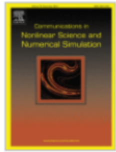

Research paper

Delay-dependent stability switches in fractional differential equations

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Due to licensing restrictions, the full text of [11] is not included in the publicly accessible part of this habilitation thesis.

Appendix G

Paper on higher-order two-term FDDE [12] (CNSNS, 2023)

Previously in [13], we studied two-term FDDE of orders less than one. It is known that the stability regions for first and second-order equations are qualitatively very different, in this case a connected unbounded set with a boundary formed by a line and a transcendental curve, in contrast to infinitely many touching triangles. Thus, in [12] (co-author: J. Čermák; my author's share 50 %), we aimed to describe the transition between first-order and second-order two-term equations and perform a comprehensive analysis of the expected stability switches.

Building on our previous experience and the proving techniques developed in earlier works, we described the region of delay-independent stability, the stability boundary, and provided a detailed analysis of stability switches. All of that in the explicit form. We included all necessary details for precise calculations of their number and values, which were then applied on the stabilization and destabilization of fractional oscillators. The equation studied in this paper represents the simplest case of real-valued problem where such a rich stability properties occur.

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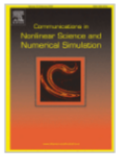

Research paper

Stabilization and destabilization of fractional oscillators via a delayed feedback control

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Due to licensing restrictions, the full text of [12] is not included in the publicly accessible part of this habilitation thesis.